(107) $r(t) = \begin{pmatrix} \frac{1}{2}\cos t \\ \frac{1}{2}\sin t \\ \sqrt{\frac{3}{4}}\, t \end{pmatrix}$

$\bar{r}'(t) = \begin{pmatrix} -\frac{1}{2}\sin t \\ \frac{1}{2}\cos t \\ \sqrt{3/4} \end{pmatrix}$

$|\bar{r}'(t)| = \sqrt{\left(\frac{1}{2}\right)^2 \sin^2 t + \left(\frac{1}{2}\right)^2 \cos^2 t + \left(\sqrt{3/4}\right)^2}$

$= \sqrt{\left(\frac{1}{2}\right)^2 + 3/4}$

$= 1 \Leftarrow \quad T(t) = \bar{r}'(t)$

$s = \int_0^{2\pi} |\bar{r}'(t)|\, dt = \int_0^{2\pi} 1 \cdot dt = 2\pi$

---

(111) $\bar{r}(t) = \begin{pmatrix} \sqrt{2}\, t \\ e^t \\ e^{-t} \end{pmatrix} \quad \bar{r}'(t) = \begin{pmatrix} \sqrt{2} \\ e^t \\ -e^{-t} \end{pmatrix}$

$|\bar{r}'(t)|^2 = 2 + e^{2t} + e^{-2t}$

$= (e^t + e^{-t})^2$

$|r'(t)| = e^t + e^{-t}$

hint: $(e^t + e^{-t})^2$

$= e^{2t} + 2e^t e^{-t} + e^{-2t}$

$= e^{2t} + 2 + e^{-2t}$

$s = \int_0^1 |\bar{r}'(t)|\, dt$

$= \int_0^1 (e^t + e^{-t})\, dt$

$= e^t - e^{-t} \Big|_0^1$

$= e - e^{-1}$

(119) $\bar{r}(t) = \begin{pmatrix} t \\ t^2 \\ t \end{pmatrix}$ find $T(t)$

$\bar{r}'(t) = \begin{pmatrix} 1 \\ 2t \\ 1 \end{pmatrix}$

velocity

$|\bar{r}'(t)| = \sqrt{2 + 4t^2}$

$\Longrightarrow T(t) = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|} = \dfrac{1}{\sqrt{2+4t^2}} \begin{pmatrix} 1 \\ 2t \\ 1 \end{pmatrix}$

unit tangent

normalize

(129) $\vec{r}(t) = \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}$

$$\vec{r}'(t) = \begin{pmatrix} e^t \sin t + e^t(+\cos t) \\ e^t \cos t + e^t(-\sin t) \end{pmatrix}$$

$$= e^t \begin{pmatrix} \sin t + \cos t \\ \cos t - \sin t \end{pmatrix}$$

$$\Rightarrow |\vec{r}'(t)| = e^t \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2}$$

$$= e^t \sqrt{(\sin^2 t + 2\sin t\cos t + \cos^2 t) + (\cos^2 t - 2\sin t \cos t + \sin^2 t)}$$

$$= e^t \sqrt{2}$$

$$s(t) = \int_0^t |\vec{r}'(t)|\,dt = \int_0^t \sqrt{2}\, e^t\,dt$$

$$= \sqrt{2}\left[e^t\right]_0^t$$

$$= \sqrt{2}(e^t - 1)$$

$$s = \sqrt{2}(e^t - 1)$$

$$\frac{s}{\sqrt{2}} = e^t - 1$$

$$e^t = \frac{s}{\sqrt{2}} + 1 \qquad \vec{r}(t) = \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}$$

$$t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

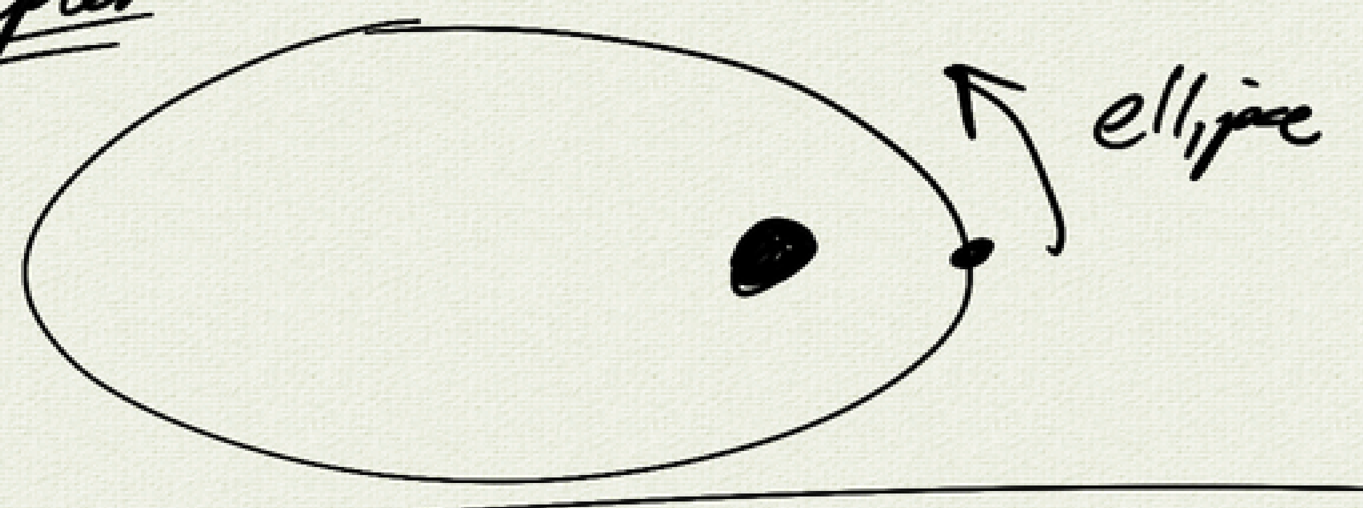
$$\Rightarrow \vec{r}(s) = \left(\frac{s}{\sqrt{2}} + 1\right) \begin{pmatrix} \sin \ln\left(\frac{s}{\sqrt{2}} + 1\right) \\ \cos \ln\left(\frac{s}{\sqrt{2}} + 1\right) \end{pmatrix}$$

---

reparametrize by arc length

1. $s(t)$ $\left(= \int_0^t |\vec{r}'(t)|\,dt \right)$
2. solve for t (in terms of s)
3. substitute to find $\vec{r}(s)$

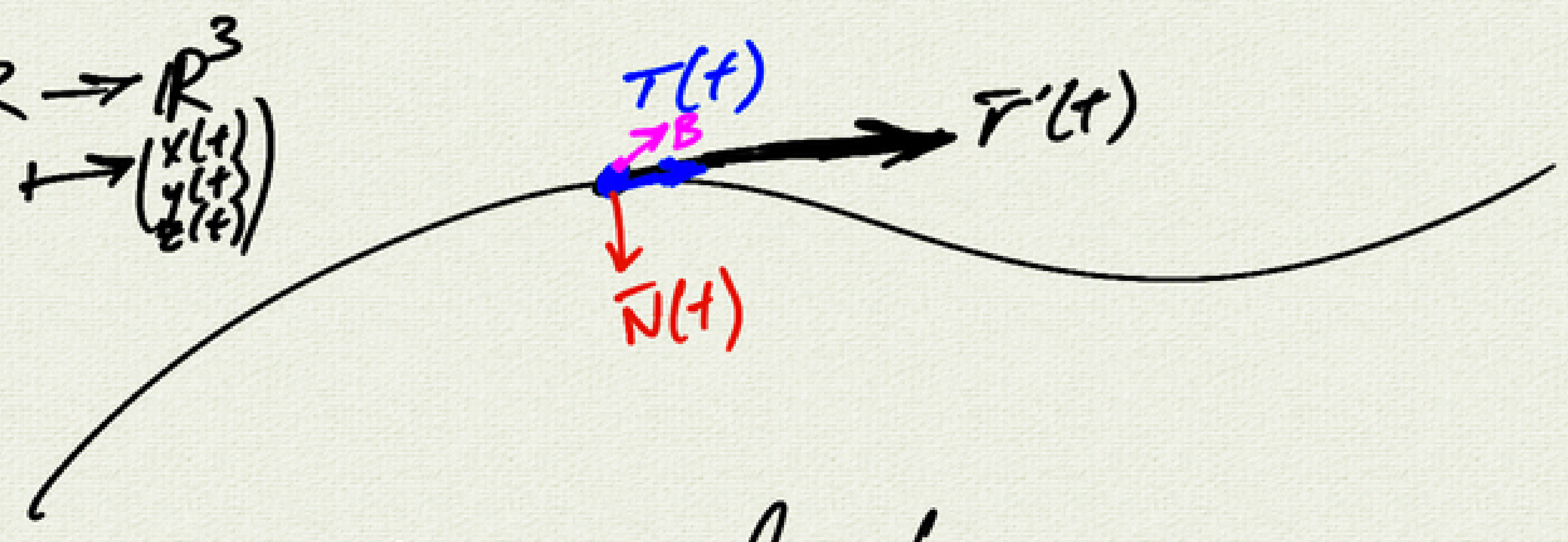
Kepler



2.4 Tangent, normal, binormal
$T \qquad N \qquad B$

$\vec{r}: \mathbb{R} \to \mathbb{R}^3$

$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$



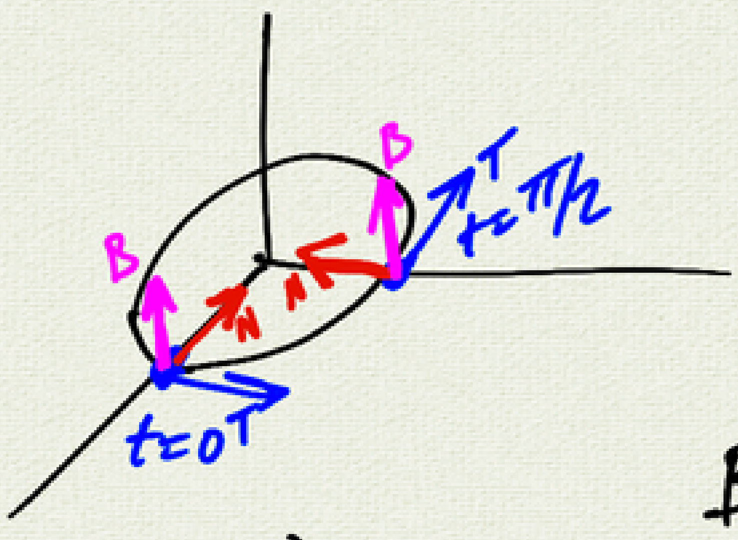
$N(t) = \dfrac{T'(t)}{|T'(t)|}$ normal vector

$B(t) = T \times N$ binormal

$= T(t) \times N(t)$

---

example: $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$ $\quad \vec{r}\,'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} = T(t)$



$T'(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} = N(t)$ (because $|T'(t)| = 1$)

$B(t) = T(t) \times N(t)$

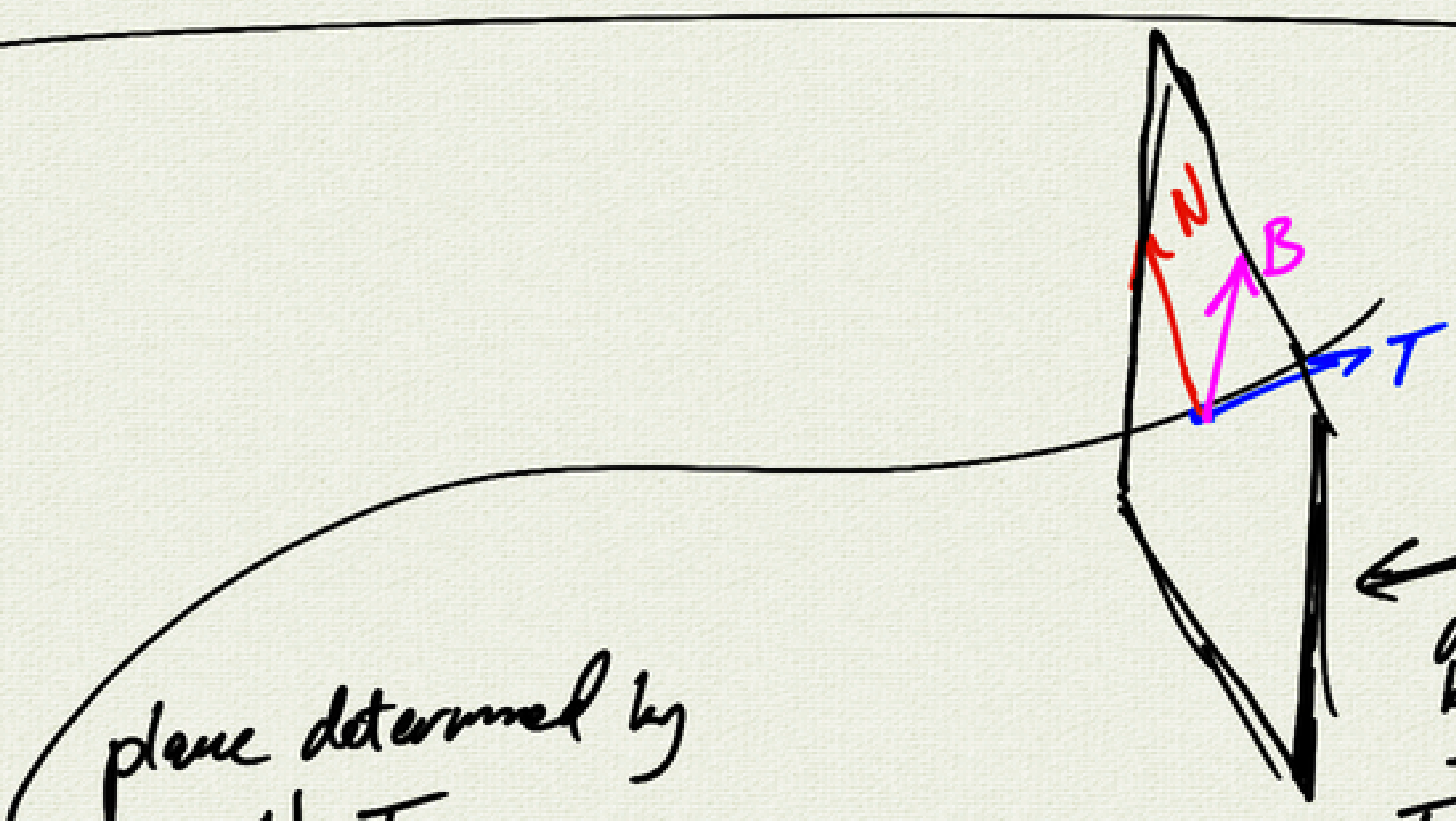
$T(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad T(\frac{\pi}{2}) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$N(0) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad N(\frac{\pi}{2}) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$B(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad B(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$= \langle 0, 0, 1 \rangle$



plane determined by N, B

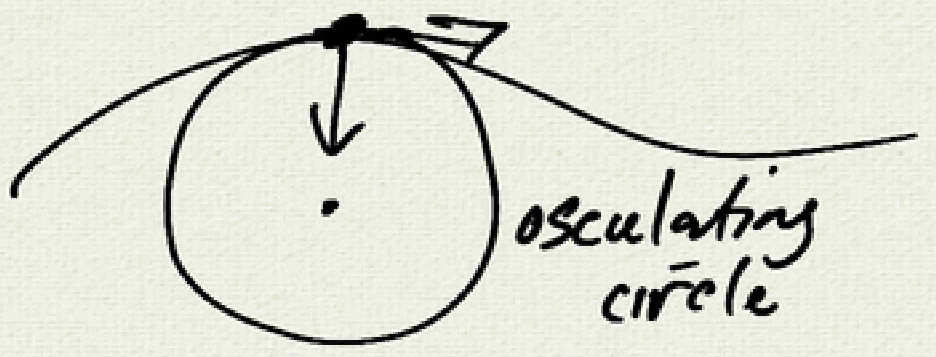
T = normal

normal plane

plane determined by N, T

osculating plane

(B = normal)



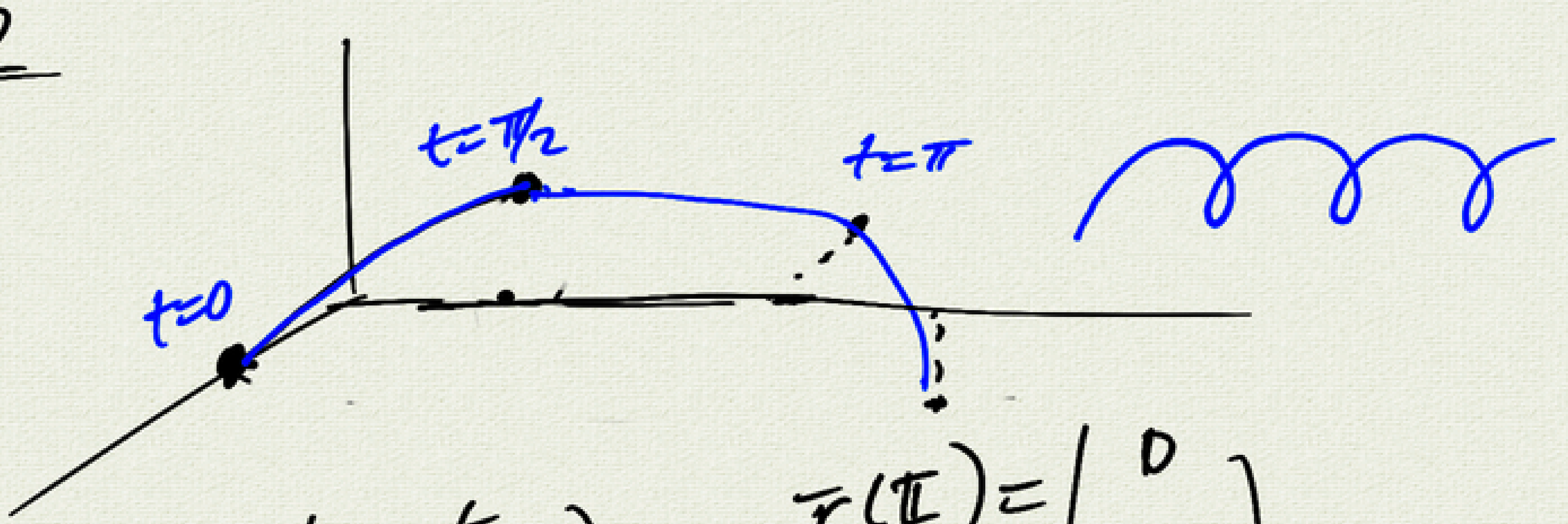
rectifying plane

determined by T, B

(N = normal)

example 2

helix



$$\bar{r}(t) = \begin{pmatrix} \cos t \\ t \\ \sin t \end{pmatrix} \qquad \bar{r}\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} 0 \\ \pi/2 \\ 1 \end{pmatrix}$$

$$\bar{r}'(t) = \begin{pmatrix} -\sin t \\ 1 \\ \cos t \end{pmatrix} \qquad |\bar{r}'(t)| = \sqrt{2}$$

$$T(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin t \\ 1 \\ \cos t \end{pmatrix}$$

$$T'(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos t \\ 0 \\ -\sin t \end{pmatrix} \qquad |T'(t)| = \frac{1}{\sqrt{2}}$$

$$\Longrightarrow N(t) = \frac{T'(t)}{|T'(t)|} = \begin{pmatrix} -\cos t \\ 0 \\ -\sin t \end{pmatrix}$$

$$B(t) = T \times N = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & 1 & \cos t \\ -\cos t & 0 & -\sin t \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin t \\ -1 \\ \cos t \end{pmatrix}$$

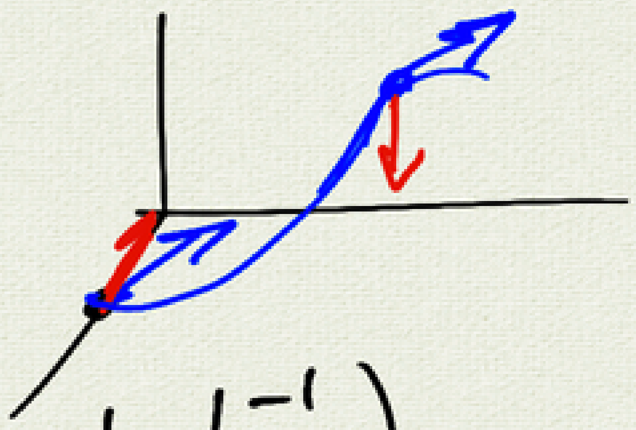
$$\bar{u} \times k\bar{v} \stackrel{?}{=} k(\bar{u} \times \bar{v})$$
$$= |\bar{u}||k\bar{v}| \sin\theta$$
$$= k\,|\bar{u}||\bar{v}| \sin\theta$$
$$= k\,|\bar{u} \times \bar{v}|$$

$t=0$

$$T(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$N(0) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$B(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$



$$T\left(\tfrac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$N\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$B\left(\tfrac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$