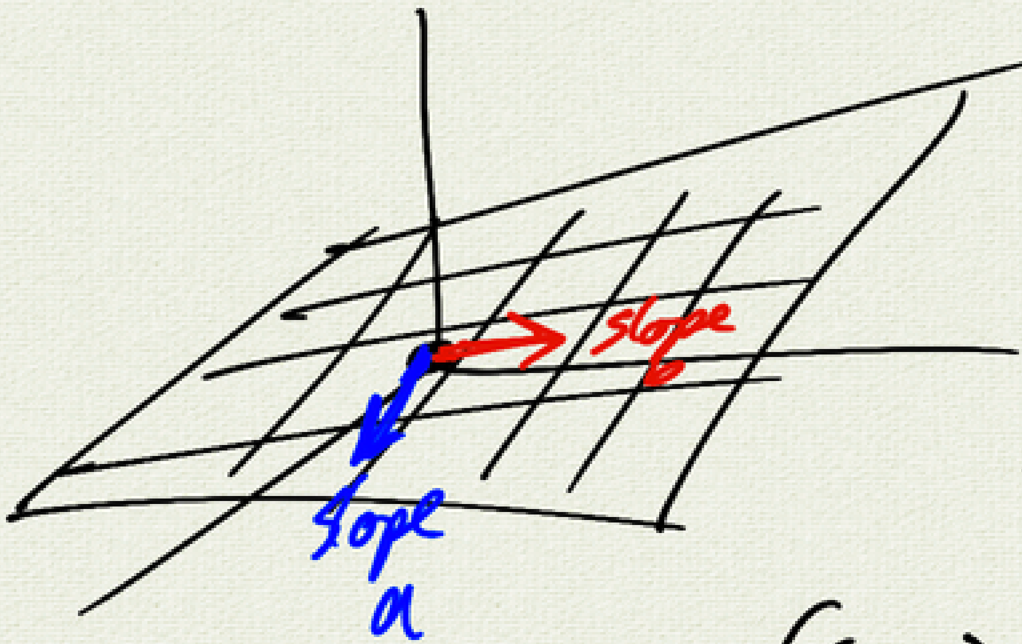
3.4 Differentiability



plane through origin

$$z = ax + by$$

slope in x-direction   slope in y-direction

$$f(x,y) = ax + by$$

$$\frac{\partial f}{\partial x} = a \qquad \frac{\partial f}{\partial y} = b$$

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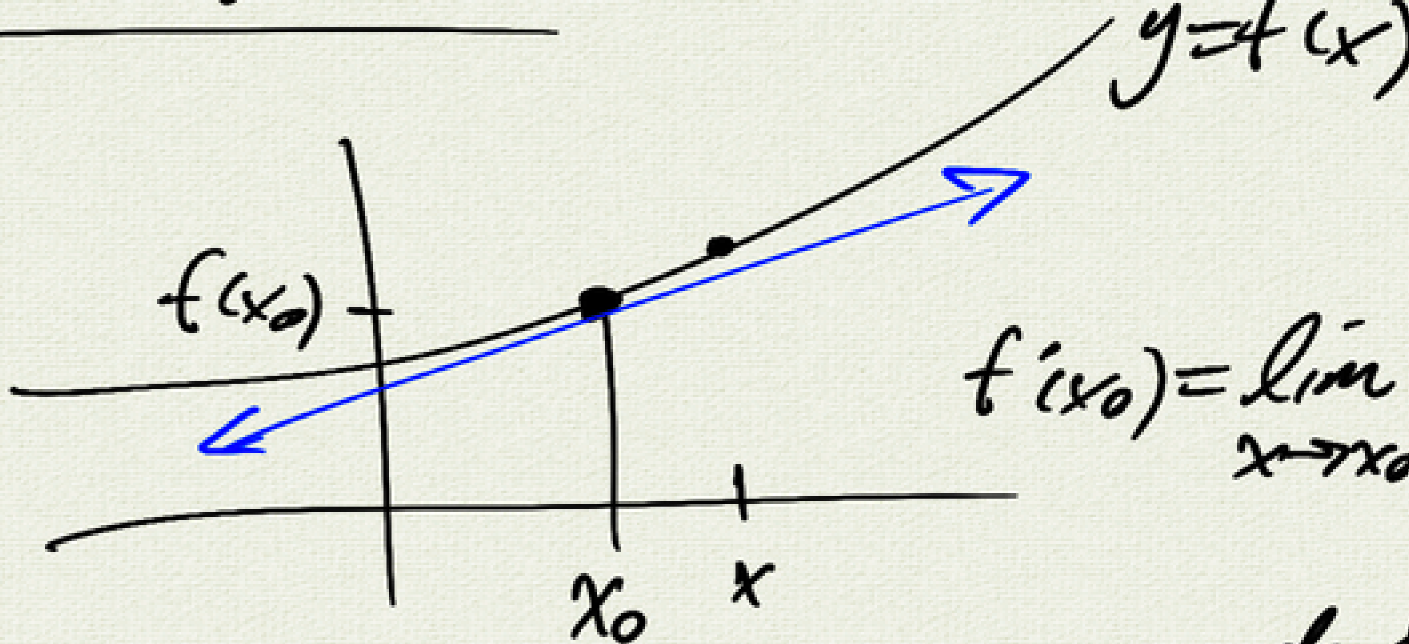
plane through (x_0, y_0, z_0):

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

$$z = z_0 + a(x - x_0) + b(y - y_0)$$

$$= z_0 + a\,\Delta x + b\,\Delta y$$

differentiability in $\mathbb{R}$:



$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

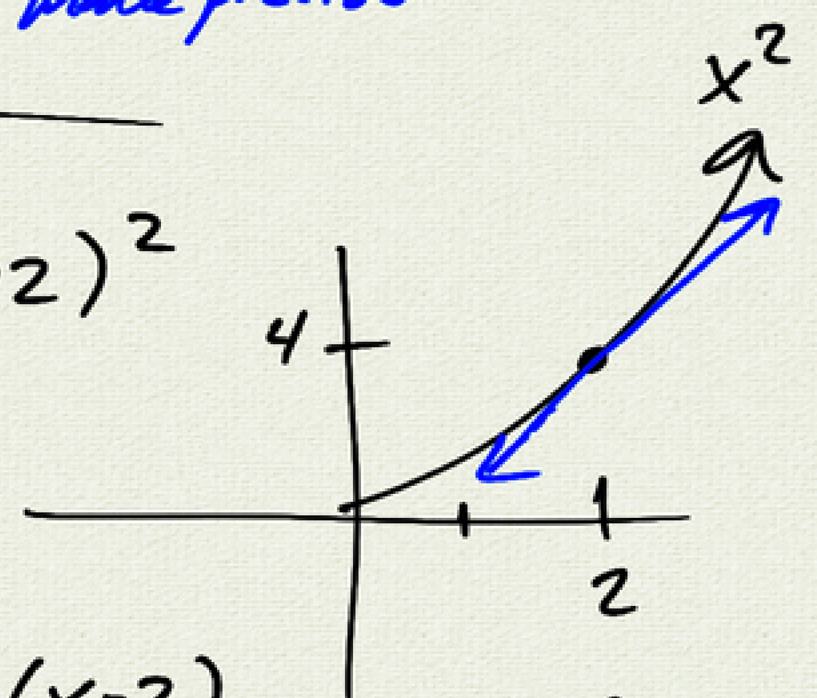
approximate f by tangent line:

$$f(x)\approx f(x_0)+f'(x_0)(x-x_0)$$

can make precise

<u>example</u>:

approximate $(2.02)^2$



$f(x)=x^2$

near $x_0=2$:

$$f(x)\approx f(2)+f'(2)(x-2)$$

$$\approx 4+4(.02)$$

$$\approx 4.08$$

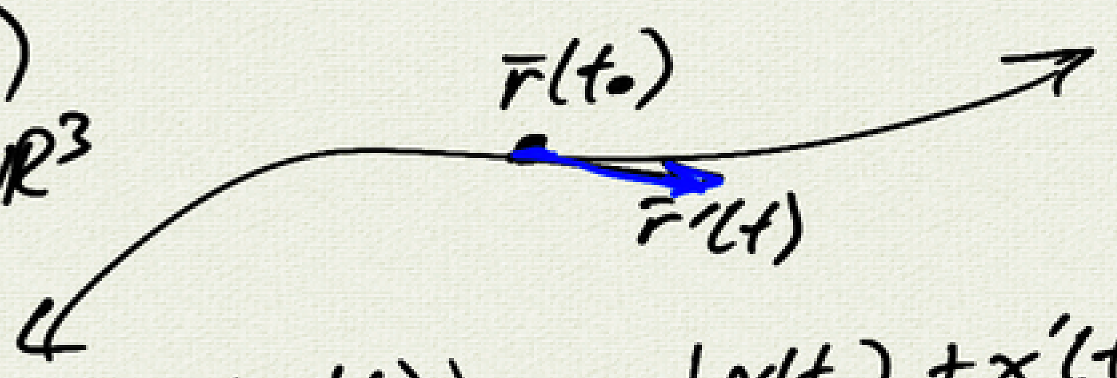
$f'(x)=2x$

$f'(2)=4$

$$f(2.02)=4.0804$$

$\bar{r}(t)$

$\bar{r} : \mathbb{R} \to \mathbb{R}^3$

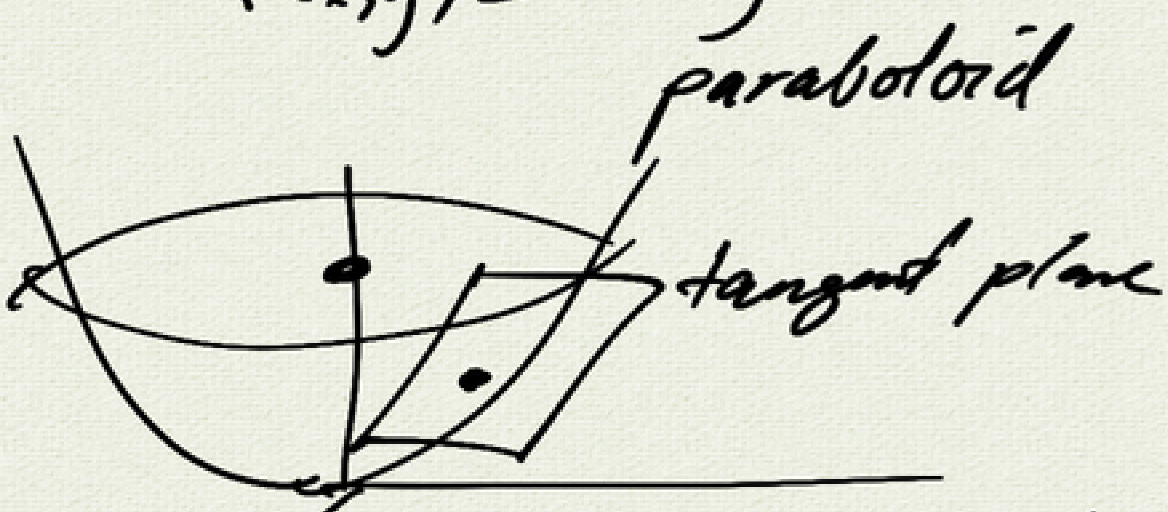


$$\bar{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \approx \begin{pmatrix} x(t_0) + x'(t_0)(t-t_0) \\ y(t_0) + y'(t_0)(t-t_0) \\ z(t_0) + z'(t_0)(t-t_0) \end{pmatrix}$$

$$\boxed{\bar{r}(t) \approx \bar{r}(t_0) + \bar{r}'(t_0)(t-t_0)}$$

$z = f(x,y)$ $\qquad f(x,y) = x^2 + y^2$

$f : \mathbb{R}^2 \to \mathbb{R}$



equation of tangent plane

$$z = z_0 + f_x(x - x_0) + f_y(y - y_0)$$

f is <u>differentiable</u> at (x_0, y_0)

if $\quad f(x,y) \approx f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$

<u>example</u> approximate $f(1.01, 1.97)$ $\qquad$ where $f(x,y) = x^2 + y^2$

"negative" from 2

$f_x = 2x$

$f_y = 2y$

$f(1,2) = 1^2 + 2^2 = 5$

$$f(x,y) \approx f(1,2) + \underset{f_x(1,2)}{f_x}(x-1) + \underset{f_y(1,2)}{f_y}(y-2)$$

$$\boxed{f(x,y) \approx 5 + 2(x-1) + 4(y-2)}$$ linear approximation near $(1,2)$

$z = 5 + 2(x-1) + 4(y-2)$

tangent plane

$$\begin{aligned} f(1.01, 1.97) &\approx 5 + 2(.01) + 4(-.03) \\ &\approx 5 + .02 - .12 \\ &\approx 4.9 \end{aligned}$$

$f(1.01, 1.97) = 4.901$

examples:

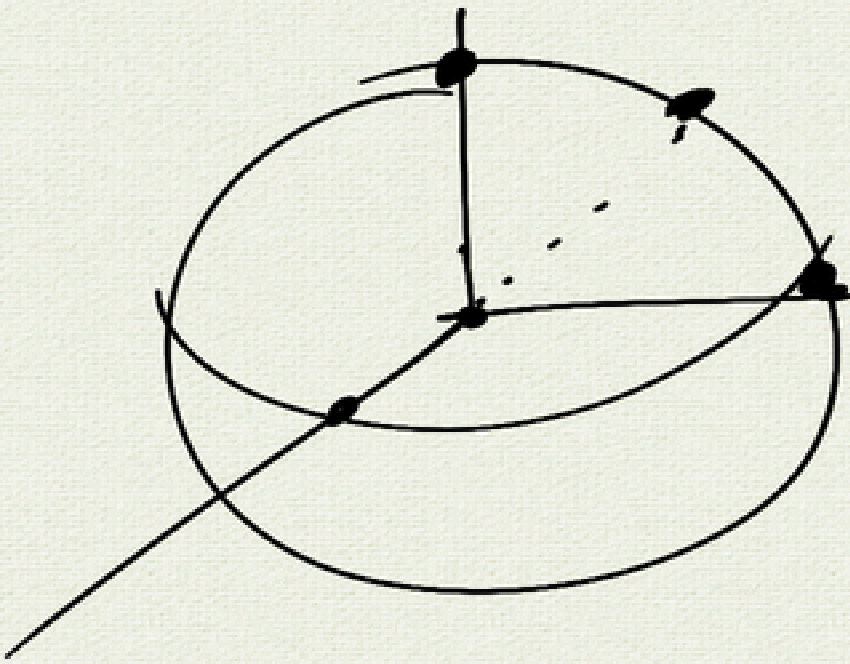
unit sphere

$$x^2+y^2+z^2=1$$

$$f(x,y)=\sqrt{1-x^2-y^2}=(1-x^2-y^2)^{1/2}$$

find tangent plane at $(0,0,1)$ and $\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

and $(0,1,0)$



$$f_x=\frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2-y^2}}$$

$$=\frac{-x}{\sqrt{1-x^2-y^2}}=-\frac{x}{z}$$

$$f_y=-\frac{y}{z}$$

tangent plane

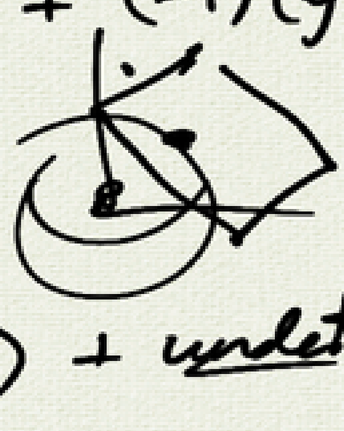
$$z=f(x_0,y_0)+f_x(x-x_0)+f_y(y-y_0)$$

$(0,0,1)$ $\quad z=1+0(x-x_0)+0(y-y_0)$

$z=1$

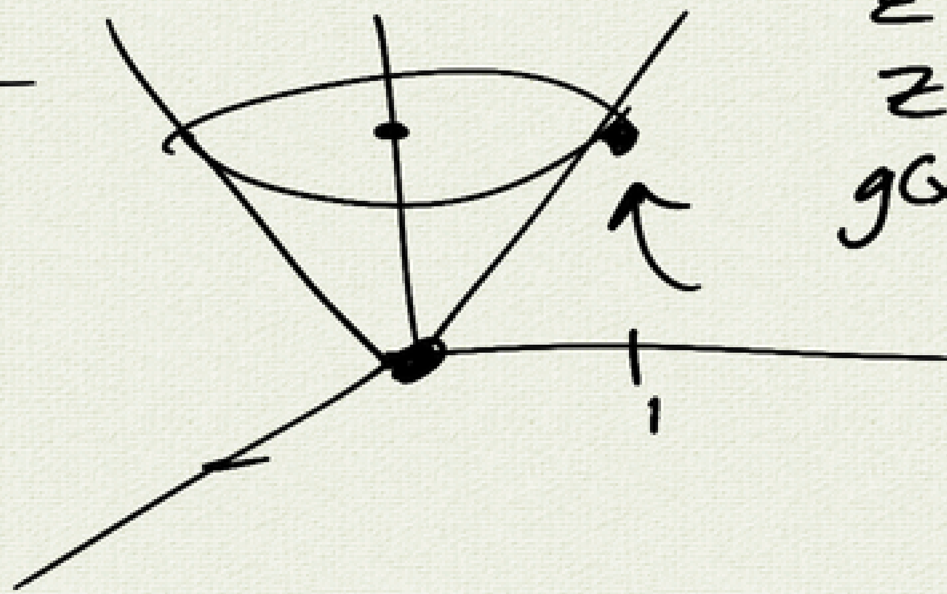
$\left(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ $\quad z=\frac{\sqrt{2}}{2}+0(x-0)+(-1)\left(y-\frac{\sqrt{2}}{2}\right)$

$y+z=\sqrt{2}$



$(0,1,0)$ $\quad z=0+$ undef$(\quad)+$ undef$(\quad)$

cone



$z = r$

$z = \sqrt{x^2+y^2}$

$g(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

find tangent plane at

$(0, 1, 1)$

$g_x = \frac{1}{2}\frac{2x}{\sqrt{(x^2+y^2)}} = \frac{x}{z} = \frac{x}{r} \Leftarrow$ undefined at $(0,0)$

$g_y = \frac{y}{r}$

(not differentiable at $(0,0)$
continuous at $(0,0)$)

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at $(x,y) = (0,1)$:

tangent plane

$$z = g(0,1) + g_x(x-x_0) + g_y(y-y_0)$$

$$= 1 + 0(x-0) + 1(y-1)$$

$$z = y$$

weird example

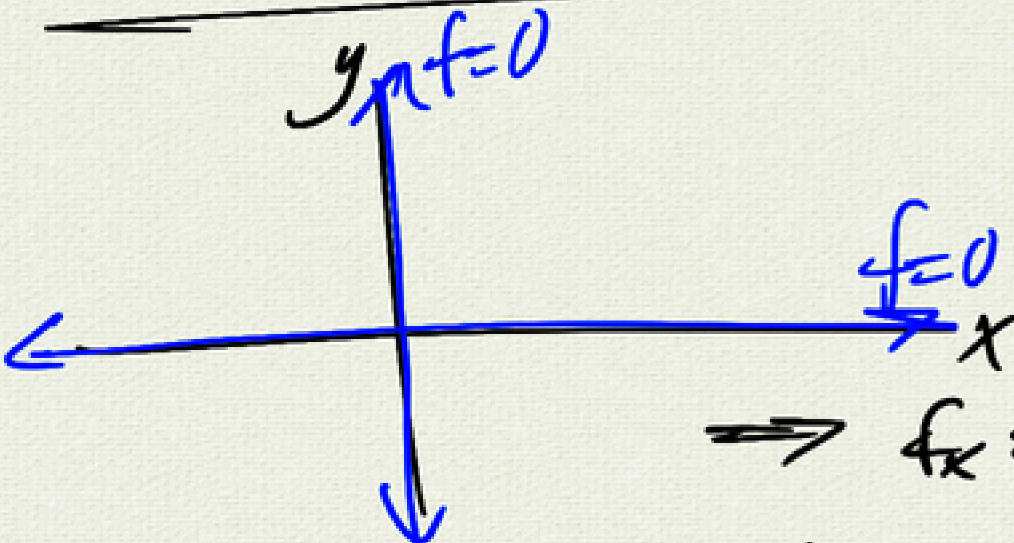
$$f(x,y) = \frac{2xy}{x^2+y^2}$$

$$\left(= \sin 2\theta\right)$$

similar to

$$\frac{x^2-y^2}{x^2+y^2} = \cos 2\theta$$

infinite ruffled collar



$x=0 \Rightarrow f=0$

$y=0 \Rightarrow f=0$

$\Rightarrow f_x = 0 = f_y = 0$ at $(0,0)$

but not differentiable

(tangent plane is not a good approximation)