195 $z = f(x,y) = xe^y$

$f_x = e^y \qquad f_x(1,2) = e^2$

$f_y = xe^y \qquad f_y(1,2) = e^2$

$\Delta z \quad (1,2)$

$(1.05, 2.1)$

approximate $f(x,y)$ near $(x_0,y_0) = (1,2)$

$$f(x,y) \approx f(1,2) + f_x(\underbrace{x - x_0}_{\Delta x}) + f_y(\underbrace{y-y_0}_{\Delta y})$$

$$\approx e^2 + e^2 \Delta x + e^2 \Delta y$$

$$f(1.05, 2.1) \approx e^2 + \underbrace{e^2(.05) + e^2(.1)}_{\text{approx change } \approx 1.10835}$$

$e^2 \approx 7.389$

$f(1.05, 2.1) \approx 8.579$

actual change

1.185

175 $xy + yz + zx = 11$ $\quad P(1,2,3)$

$$z(x+y) = 11 - xy$$

$$f(x,y) = z = \boxed{\frac{11-xy}{x+y}}$$

$$f_x = \frac{(-y)(x+y) - (11-xy)(1)}{(x+y)^2}$$

$$= \frac{\underline{-yx} - y^2 - 11 + \underline{xy}}{(x+y)^2}$$

$$= \frac{-y^2-11}{(x+y)^2}$$

$$f_y = \frac{-x^2-11}{(x+y)^2}$$

$$f_x(1,2) = \frac{-15}{9} = -\frac{5}{3}$$

$$f_y(1,2) = \frac{-12}{9} = -\frac{4}{3}$$

tangent plane

$$z = z_0 + f_x(x-x_0) + f_y(y-y_0)$$

$$= 3 + \left(-\frac{5}{3}\right)(x-1) + \left(-\frac{4}{3}\right)(y-2)$$

$$3z = 9 - 5x + 5 - 4y + 8$$

$$\boxed{5x + 4y + 3z = 22}$$

$$x^2+y^2+z^2=r^2$$

$$\underbrace{\underbrace{3^2+4^2}_{5^2}+12^2}=13^2$$

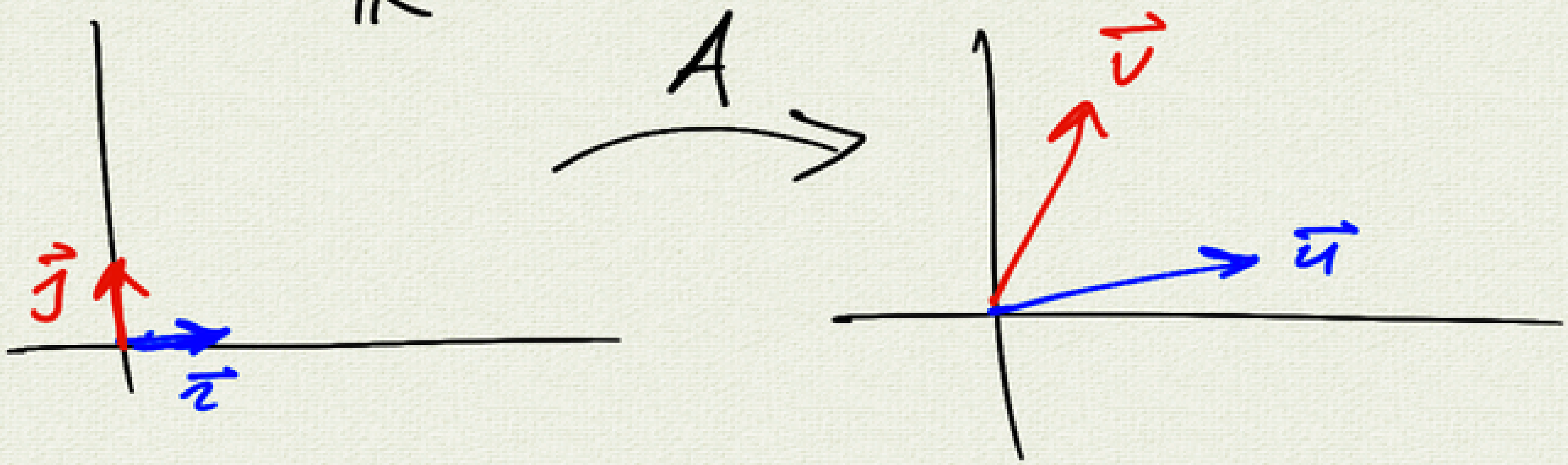
$$(3,4,12,13)$$

$$(3,4,5)$$

$$(5,12,13)$$

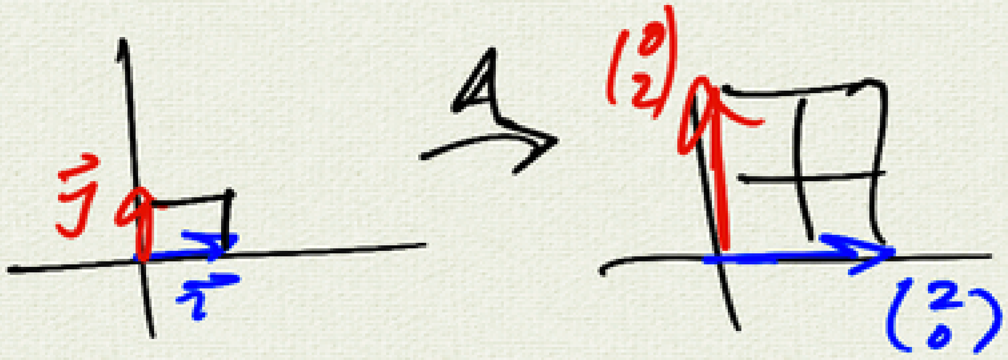
3.5 Linear transformations

$\mathbb{R}^2$



<u>scale ×2</u>

$$A = \begin{array}{c} \begin{array}{cc} \vec{u} & \vec{v} \end{array} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$



area is scaled by $\det A = 4$

apply transformation:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$A\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

matrix multiplication

A^{-1} exists $\Longleftrightarrow \det A \neq 0$

$$A = 2I$$

$$A^{-1} = \tfrac{1}{2}I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\det A^{-1} = \frac{1}{4}$$

$T : X \to Y$ $\qquad X, Y : \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$

T is a <u>linear transformation</u> if

$$T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$$

$$T(k\bar{u}) = k\,T(\bar{u})$$

---

e.g. $X = \mathbb{R}^2$ $\qquad T : \mathbb{R}^2 \to \mathbb{R}^2$

$Y = \mathbb{R}^2$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T(x\vec{i} + y\vec{j}) = x\,\overbrace{T(\vec{i})}^{\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}} + y\,\overbrace{T(\vec{j})}^{\vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}}$$

$$= x\begin{pmatrix} a \\ b \end{pmatrix} + y\begin{pmatrix} c \\ d \end{pmatrix}$$

apply transformation

$$= \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$$

matrix multiplication

$T : X \to Y$

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \quad \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$

---

last example: T = scale $\times 2$

$X = \mathbb{R}^2 \quad Y = \mathbb{R}^2$

$$\underset{\substack{A \\ 2\times 2}}{\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}} \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

---

general

$$\begin{array}{c} \\ m \text{ rows} \end{array} \underset{\substack{A \\ m\times n}}{\overset{n}{\begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{pmatrix}}} \begin{pmatrix} | \end{pmatrix} n = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \Big]\, m$$

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

---

example: $C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad$ 3×1 rows $\times$ cols

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (t) = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}$$

$$\mathbb{R} \longrightarrow \mathbb{R}^3$$

parametric line
Thru origin $\quad \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

---

example: $D = (1 \ 0 \ 0)$

$$(1 \ 0 \ 0) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x) = x$$

projection on x-axis

---

apply linear transformation $\longleftrightarrow$ matrix multiplication

$T \qquad A$

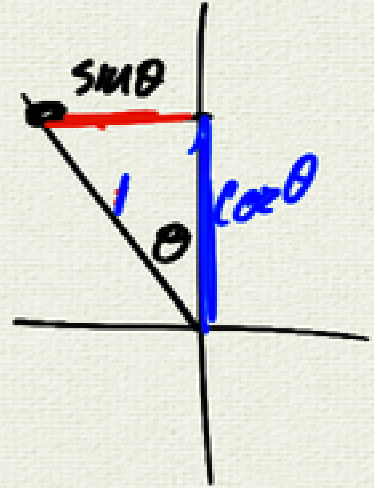
$S \qquad B$

composition $\longleftrightarrow$ matrix multiplication

$ST \qquad BA$

R_θ = rotation by θ (in $\mathbb{R}^2$)

$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$



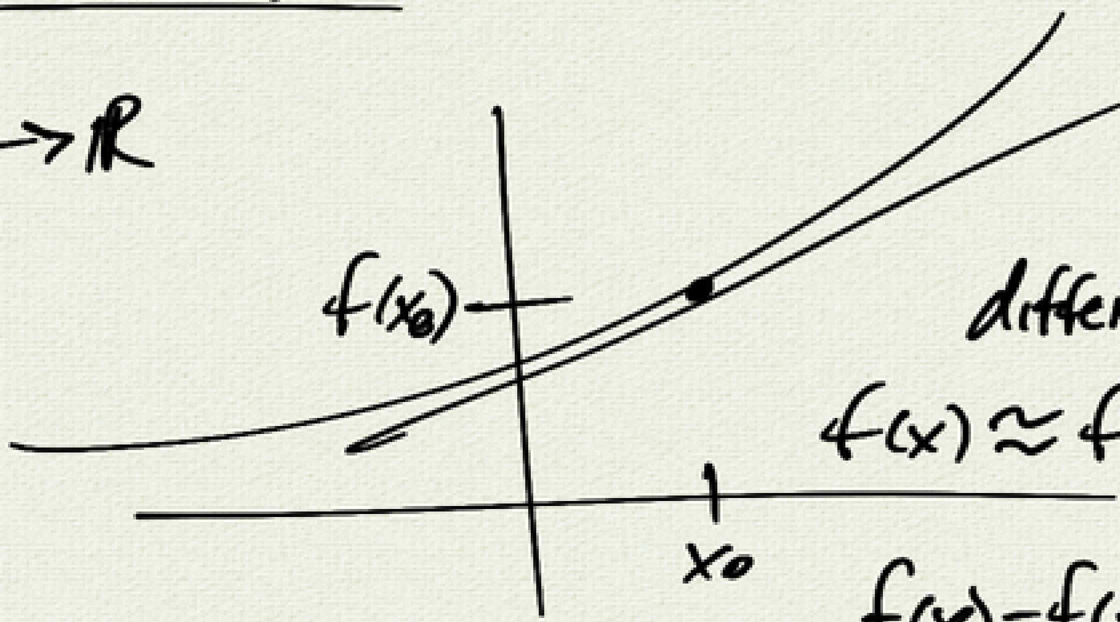
example: $R_{\pi/6}\, R_{\pi/3} = R_{\pi/2}$

verify

$$\begin{pmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1/2 & -\frac{\sqrt{3}}{2} \\ \sqrt{3}/2 & 1/2\end{pmatrix} = \begin{pmatrix}0 & -1 \\ \boxed{1} & 0\end{pmatrix} = R_{\pi/2} \checkmark$$

differentiability

$f : \mathbb{R} \to \mathbb{R}$



differentiable:

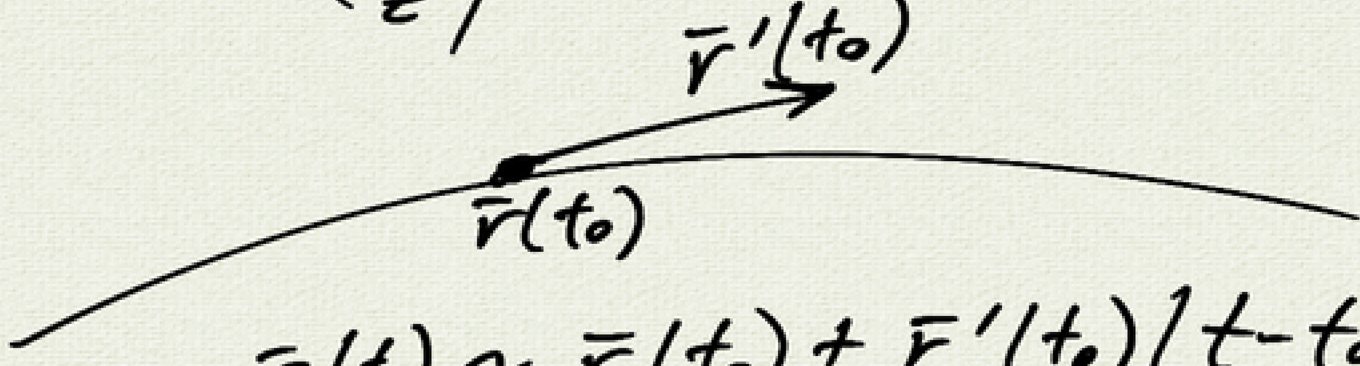
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$$

$$\Delta f \approx \underbrace{f'(x_0)}_{\substack{1\times 1 \\ \text{matrix}}} \Delta x$$

$\bar{r} : \mathbb{R} \to \mathbb{R}^3$ curve

$$t \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



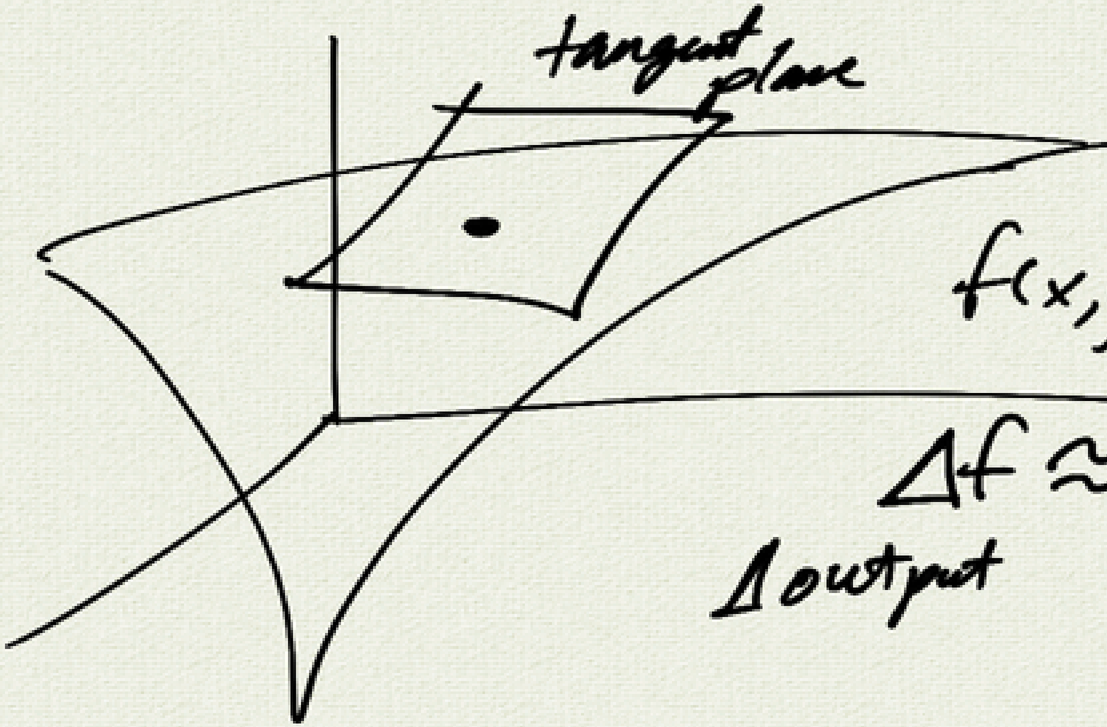
$$\bar{r}(t) \approx \bar{r}(t_0) + \bar{r}'(t_0)(t - t_0)$$

$$\bar{r}(t) - \bar{r}(t_0) \approx \bar{r}'(t_0)(t - t_0)$$

$$\Delta \bar{r} \approx \underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{\substack{3\times 1 \\ \text{linear transformation}}} \Delta t$$

$z = f(x,y)$ $\qquad \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\qquad$ surface

$\qquad\qquad (x,y) \longmapsto z$



$$f(x,y) \approx f(x_0,y_0) + f_x(\underbrace{x-x_0}_{\Delta x}) + f_y(\underbrace{y-y_0}_{\Delta y})$$

$$\underset{\Delta\,\text{output}}{\Delta f} \approx \underbrace{\overbrace{\begin{pmatrix} f_x & f_y \end{pmatrix}}^{df}}_{\substack{1\times 2 \\ \text{linear} \\ \text{transformation}}} \underset{\Delta\,\text{input}}{\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}}$$

---

composition $\longrightarrow$ chain rule