$$S^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 \qquad S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Scale ×2

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$ ← scale ×4

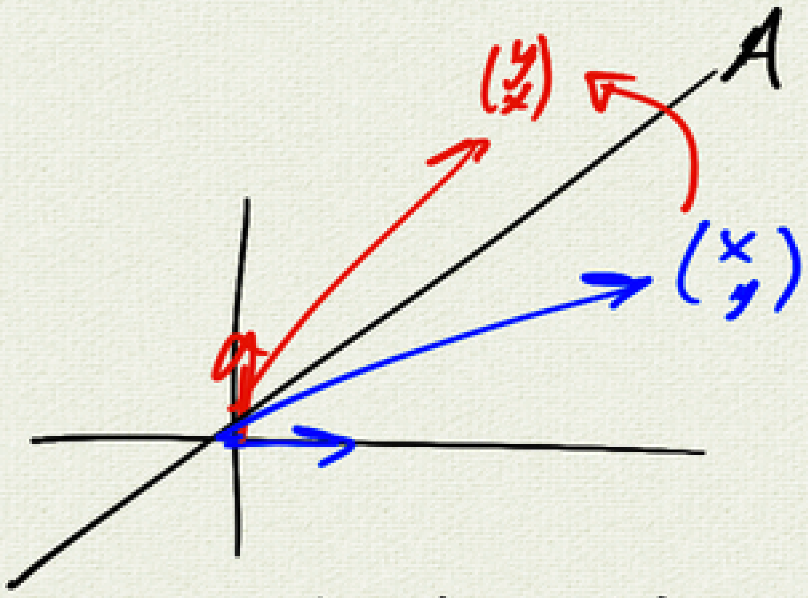
$$R_{\pi/6} = \begin{pmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{pmatrix} \Longleftarrow$$

$$R_{\pi/6}^2 = R_{\pi/3}$$

$$R_{\pi/6}^3 = R_{\pi/2}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = I$$



$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y$$ projection on y-axis

$$C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(t) \qquad t \longmapsto \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix} \; r(t)$$

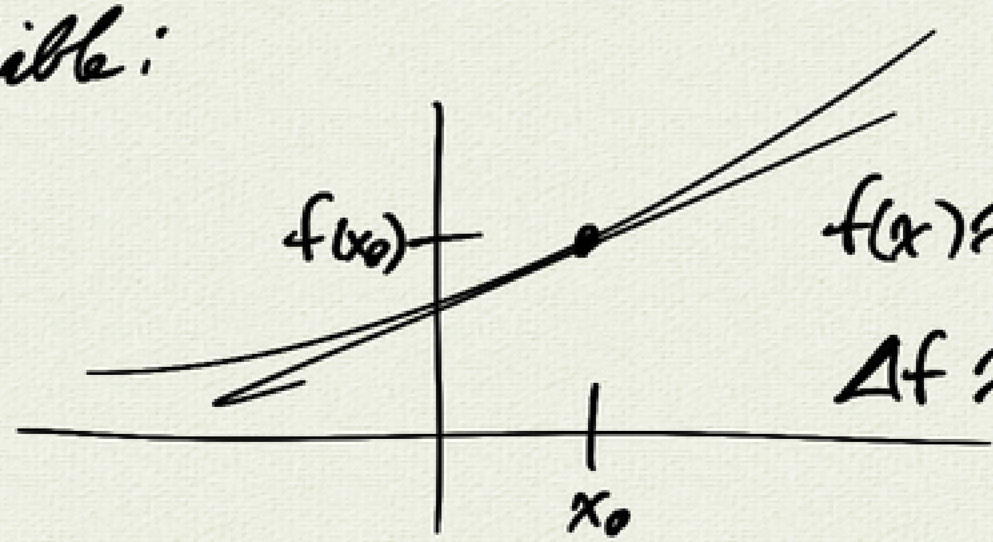
curve: straight line through origin

$$PC = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2)$$

↗ matrix multiplication = composition

3.6 Chain rule

differentiable:

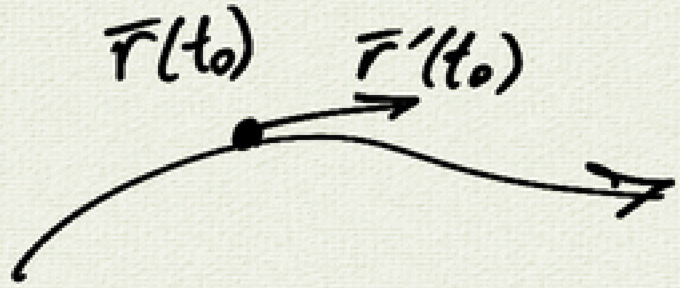


$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$\Delta f \approx f'(x_0)\Delta x$$

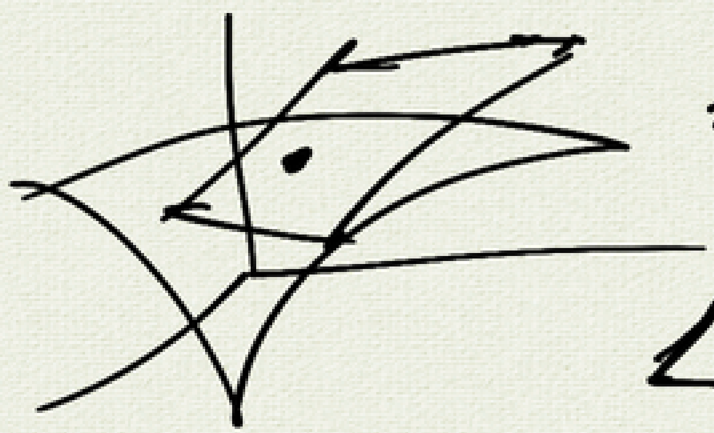
$\bar{r}: \mathbb{R} \to \mathbb{R}^3$

$t \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$$\bar{r}(t) \approx \bar{r}(t_0) + \bar{r}'(t_0)(\Delta t)$$

$$\bar{\Delta} r \approx \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Delta t$$



$$z = f(x,y)$$

$$f(x,y) \approx f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\Delta f \approx \underbrace{\begin{pmatrix} f_x & f_y \end{pmatrix}}_{df} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

composition ⟹ chain rule

4 boxes/hr

10 problems/box

time → eat nerds → # math problems

40 problems/hr

chain rule = multiplication

$$(f\circ g)'(x) = f'(\underline{g(x)}) \cdot g'(x)$$

multiplication

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$z = f(x,y)$
surface

$$z = z(t)$$
$$\Rightarrow z'(t) = ?$$

$$\Delta z \approx \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$
$$\approx f_x \Delta x + f_y \Delta y$$
$$\approx f_x \, x'(t) \Delta t + f_y \, y'(t) \Delta t$$
$$\Delta z \approx \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) \Delta t$$

$f'(t)$ →

chain rule:

$$f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

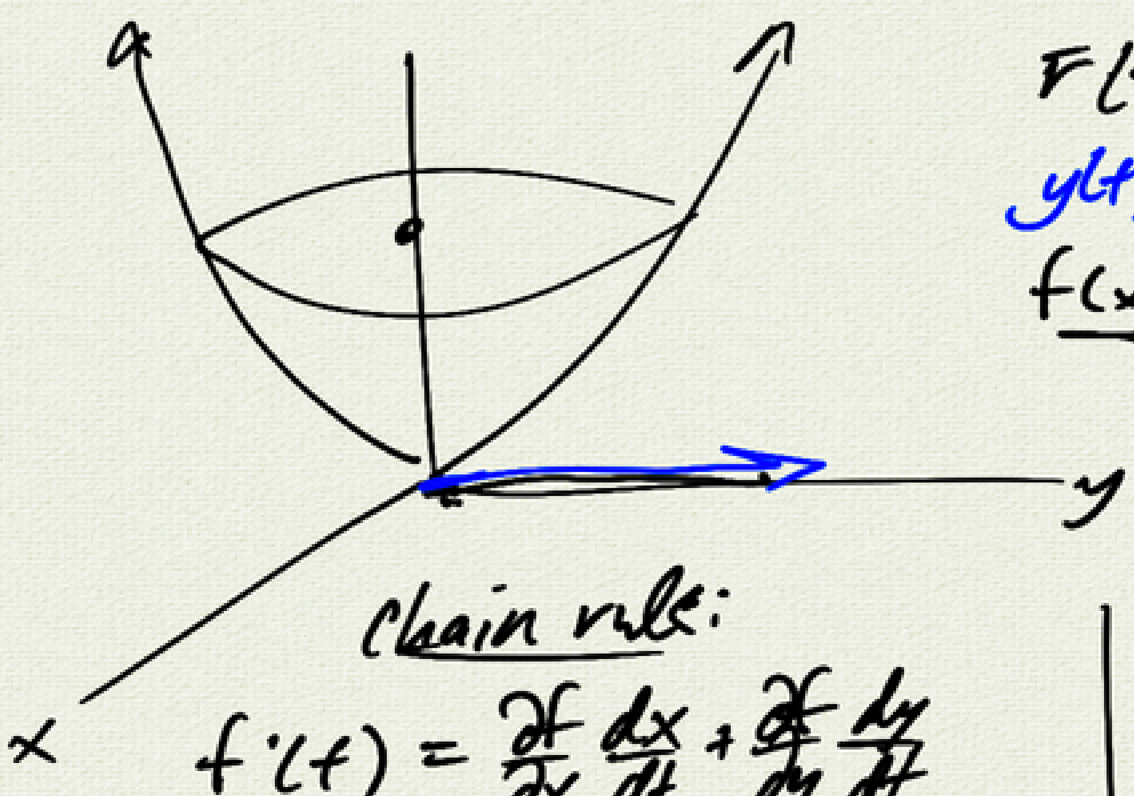
another look:

$$\Delta f \approx \left(\frac{\partial f}{\partial x} \; \frac{\partial f}{\partial y} \right) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$
$$\approx \left(\frac{\partial f}{\partial x} \; \frac{\partial f}{\partial y} \right) \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} \Delta t$$

$f'(t)$

Example:

$f(x,y) = x^2 + y^2$ paraboloid



$x(t)$, $y(t)$

$\bar{r}(t) = \begin{pmatrix} 0 \\ t \end{pmatrix} \Rightarrow \bar{r}'(t) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$f_x = 2x$

$f_y = 2y$

$f(x(t), y(t))$

calculate $f'(t)$:

chain rule:

$$f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= 2x(0) + 2y(1)$$
$$= 2y$$
$$= 2t$$

$f(t) = x(t)^2 + y(t)^2$
$= t^2$

$\rightarrow f'(t) = 2t$

---

another curve, same surface

$\bar{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

$f(x,y) = x^2 + y^2$

$\bar{r}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

$f_x = 2x$

$f_y = 2y$

$$f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= 2x(1) + 2y(2t)$$
$$= 2t + 2t^2 \cdot 2t$$
$$= 2t + 4t^3$$

check: $f(x(t), y(t)) = t^2 + (t^2)^2$

$f(t) = t^2 + t^4$

$\rightarrow f'(t) = 2t + 4t^3$ ✓

$f(x,y,z)$ $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

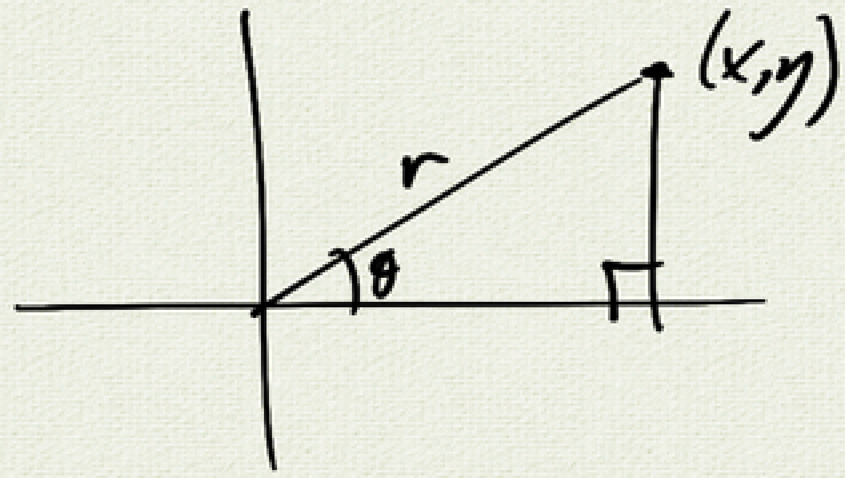
$$\rightarrow \boxed{f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

$$= \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}}_{df} \underbrace{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}}_{\bar{r}'(t)}$$

same $f(x,y)=x^2+y^2$

polar: $x = x(r,\theta) = r\cos\theta$

$y = y(r,\theta) = r\sin\theta$



$\Rightarrow f = f(r,\theta)$

$$\Rightarrow \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

also chain rule

$f(x,y)=x^2+y^2$

$f_x = 2x$

$f_y = 2y$

$x = r\cos\theta$ $\quad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\quad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$= 2x(\cos\theta) + 2y(\sin\theta)$

$= 2r\cos\theta \cdot \cos\theta + 2r\sin\theta\sin\theta$

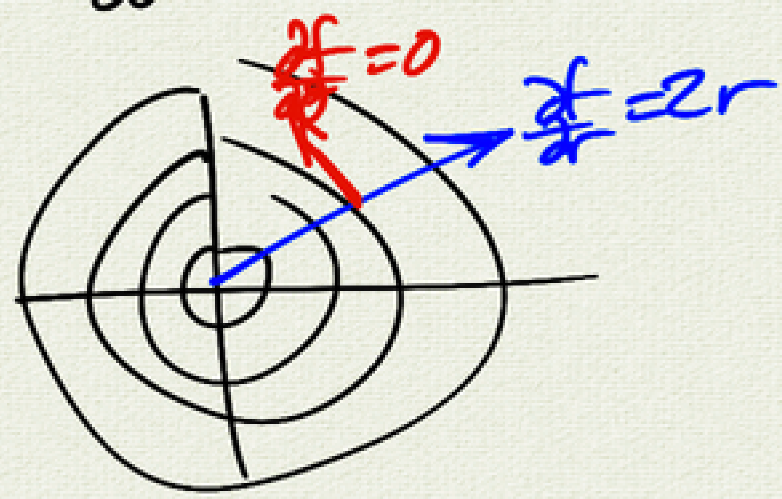
$= 2r$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$= 2x(-r\sin\theta) + 2y(r\cos\theta)$

$= 2r\cos\theta(-r\sin\theta) + 2r\sin\theta(r\cos\theta)$

$= 0$



$f(x(r,\theta), y(r,\theta)) = f(r,\theta)$

$= (r\cos\theta)^2 + (r\sin\theta)^2$

$f(r,\theta) = r^2 \Longrightarrow \frac{\partial f}{\partial r} = 2r, \ \frac{\partial f}{\partial \theta} = 0$