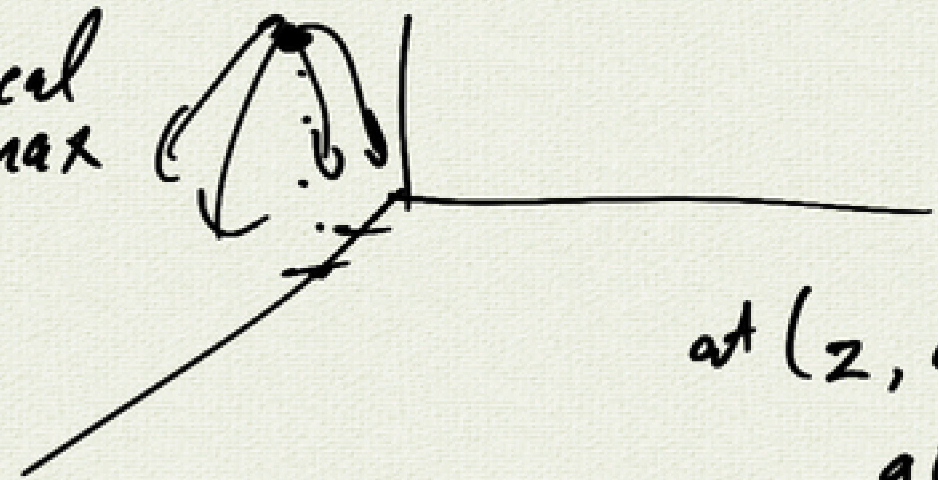
#2

$$g(x,y) = -4(x-2)^2 - 5(y^2) + 5$$

local max



$$d^2f = \begin{pmatrix} -8 & 0 \\ 0 & -10 \end{pmatrix}$$

$$\det d^2f = 80$$

at $(2,0)$

$$g(x,y) \approx 5 + \underbrace{f_x}_{0}(x-2) + \underbrace{f_y}_{0}(y-0)$$

$$\left.\begin{aligned} g_x &= -8(x-2) \\ g_y &= -10y \end{aligned}\right\} \text{critical pts when}$$

$g_x = 0 = g_y$

$x = 2,\ y = 0$

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③

$$h(x,y) = (x-1)^2 - 2(y-2)^2 - 2$$

$h_x = 2(x-1)$

$h_y = -4(y-2)$

critical pt $(1,2)$

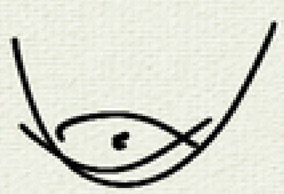
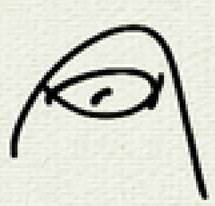
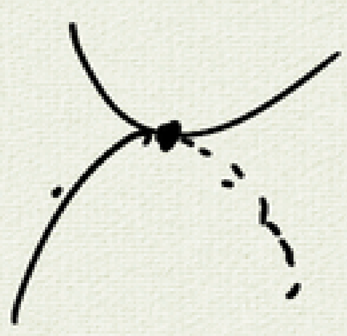
$$d^2f = \begin{pmatrix} \overset{h_{xx}}{2} & 0 \\ 0 & \underset{h_{yy}}{-4} \end{pmatrix}$$

concave up (in x direction)

concave down

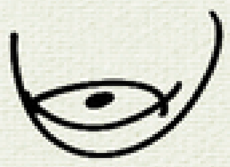
$$\det d^2f = -8$$

2nd derivative test

		d^2f	$\det d^2f$
	local min	$\begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$	$+$
	local max	$\begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}$	$+$
	saddle point	$\begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$ or $\begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix}$	$-$

4.1 Rotation of axes

$f(x,y) = Ax^2 + Bxy + Cy^2$ general quadratic

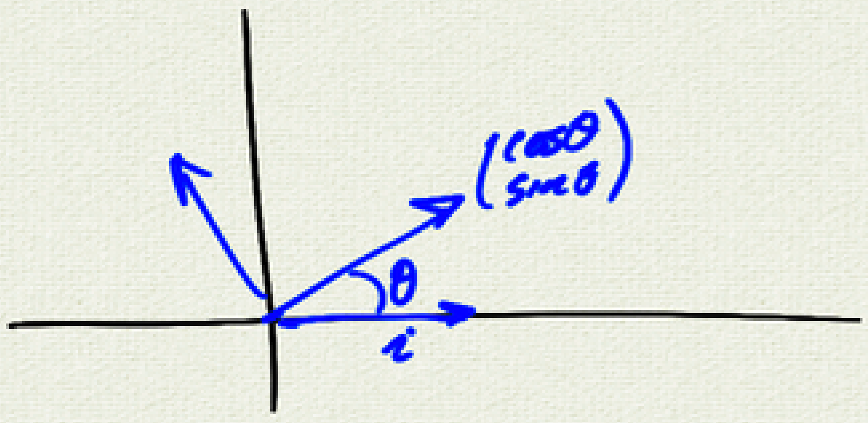
examples: $x^2 + y^2$  elliptic paraboloid

$x^2 - y^2$  saddle / pringle

$f(x,y) = Ax^2 + Bxy + Cy^2$

level sets:
ellipses
hyperbolas

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$



$\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x' \\ y' \end{pmatrix}$ $\Leftarrow$ can we choose θ to get rid of xy term?

$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

$= \begin{pmatrix} cx' - sy' \\ sx' + cy' \end{pmatrix}$

$f(x,y) = Ax^2 + Bxy + Cy^2$

$= A(cx' - sy')^2 + B(cx' - sy')(sx' + cy') + C(sx' + cy')^2$

$= \square x'^2 + \square y'^2 + \underbrace{\square}_{\text{important}} x'y'$

$[A(-2cs) + B(c^2 - s^2) + C(2sc)]\,x'y'$

$[(C-A)\,2sc + B\,(c^2 - s^2)]$

$\boxed{(C-A)\sin 2\theta + B\,(\cos 2\theta)}$ $\leftarrow$ want $= 0$

$2\sin\theta\cos\theta = \sin 2\theta$
$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$(C-A)\sin 2\theta + B\cos 2\theta = 0$

$(A-C)\sin 2\theta = B\cos 2\theta$

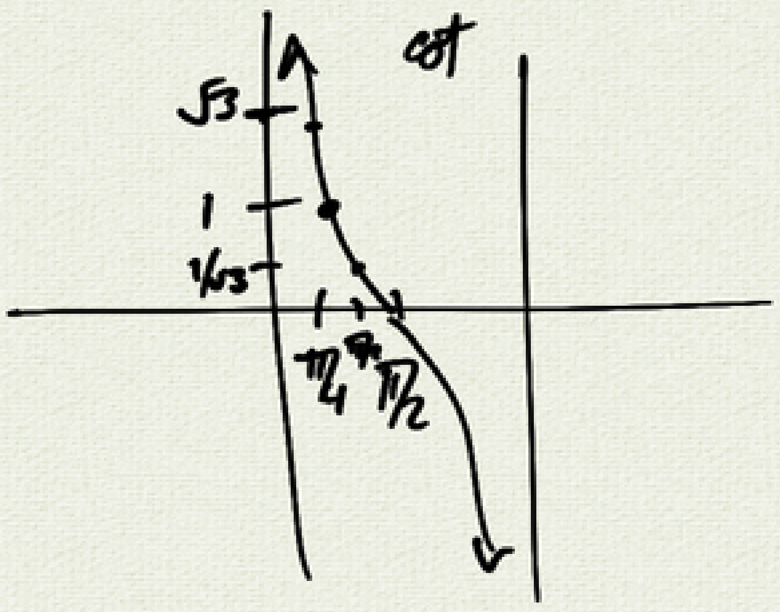
$\frac{\sin 2\theta}{\cos 2\theta} = \frac{B}{A-C} \Rightarrow$ $\boxed{\tan 2\theta = \frac{B}{A-C} \text{ or } \cot 2\theta = \frac{A-C}{B}}$

example:

$$h(x,y) = 5x^2 \underbrace{-2\sqrt{3}\,xy}_{\text{remove}} + 7y^2$$

$$5x^2 \underbrace{-2\sqrt{3}xy}_{\text{remove xy term}} + 7y^2 = 1$$

$$\cot 2\theta = \frac{A-C}{B} = \frac{5-7}{-2\sqrt{3}} = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}}$$



$$\Rightarrow 2\theta = \pi/3$$
$$\theta = \pi/6$$

$$\Rightarrow R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

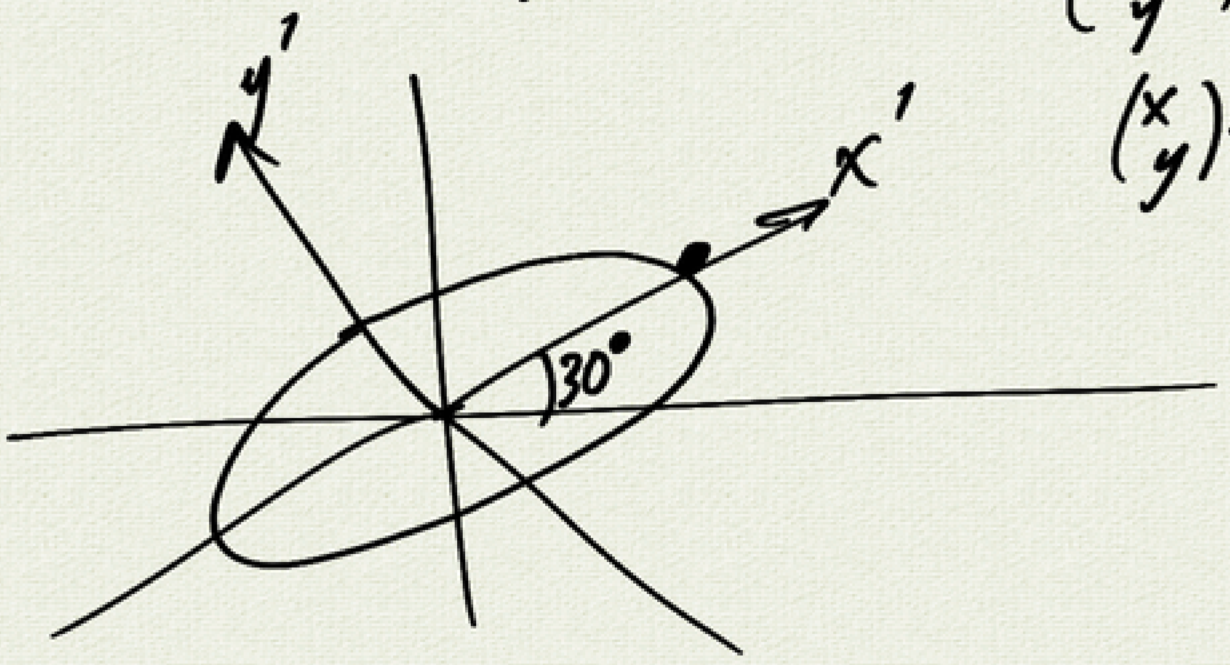
$$\left.\begin{aligned} x &= \sqrt{3}/2\, x' - \tfrac{1}{2} y' \\ y &= 1/2\, x' + \sqrt{3}/2\, y' \end{aligned}\right] \text{ substitution}$$

$$\begin{aligned} h(x,y) &= 5x^2 - 2\sqrt{3}xy + 7y^2 \\ &= 5\left(\tfrac{\sqrt{3}}{2}x' - \tfrac{1}{2}y'\right)^2 - 2\sqrt{3}\left(\tfrac{\sqrt{3}}{2}x' - \tfrac{1}{2}y'\right)\left(\tfrac{1}{2}x' + \sqrt{3}/2\, y'\right) \\ &\quad + 7\left(\tfrac{1}{2}x' + \tfrac{\sqrt{3}}{2}y'\right)^2 \\ &= 5\left(\tfrac{3}{4}x'^2 - \tfrac{\sqrt{3}}{2}x'y' + \tfrac{1}{4}y'^2\right) - 2\sqrt{3}\left(\tfrac{\sqrt{3}}{4}x'^2 + \tfrac{1}{2}x'y' - \tfrac{\sqrt{3}}{4}y'^2\right) \\ &\quad + 7\left(\tfrac{1}{4}x'^2 + \tfrac{\sqrt{3}}{2}x'y' + \tfrac{3}{4}y'^2\right) \\ &= x'^2\underbrace{\left(5\cdot\tfrac{3}{4} - \tfrac{2\cdot 3}{4} + \tfrac{7}{4}\right)}_{4} + x'y'\underbrace{\left(-\tfrac{5\sqrt{3}}{2} - \tfrac{2\sqrt{3}}{2} + 7\tfrac{\sqrt{3}}{2}\right)}_{0} \\ &\quad + y'^2\underbrace{\left(\tfrac{5}{4} + \tfrac{2\cdot 3}{4} + \tfrac{7\cdot 3}{4}\right)}_{8} \\ &= 4x'^2 + 8y'^2 \end{aligned}$$

elliptic paraboloid

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

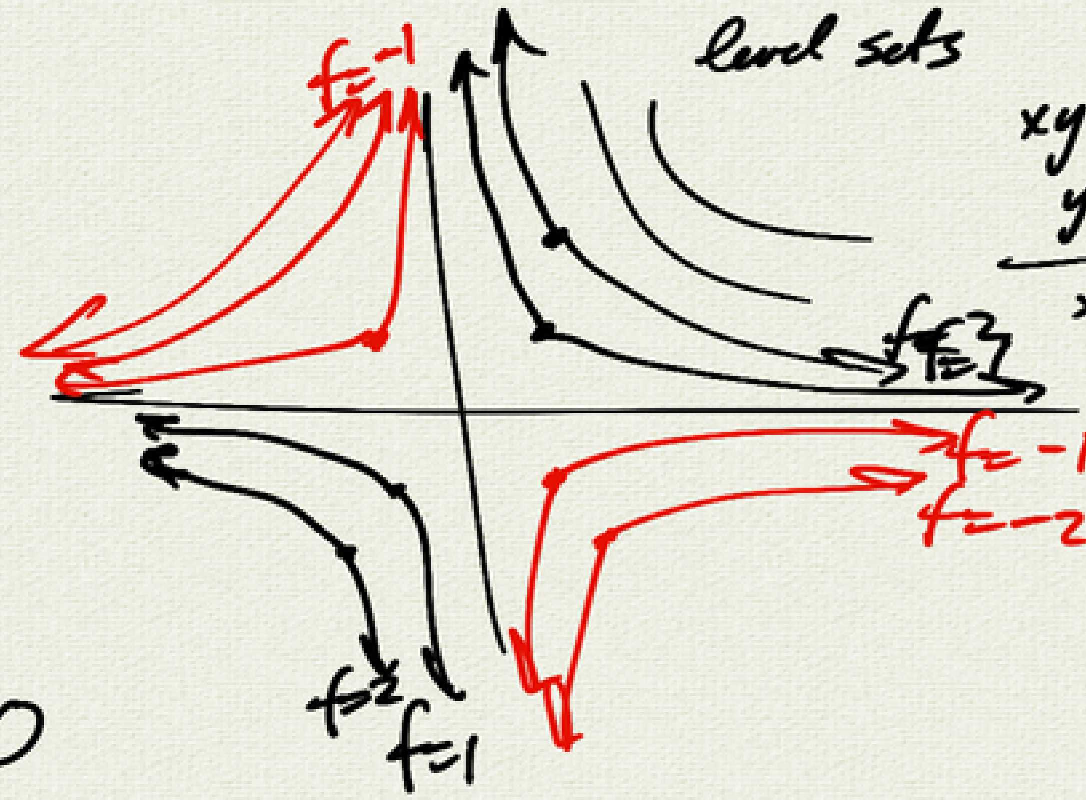
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} 5\cos 30^\circ \\ 5\sin 30^\circ \end{matrix}$$



example 2

$$f(x,y) = xy$$

rotate to remove



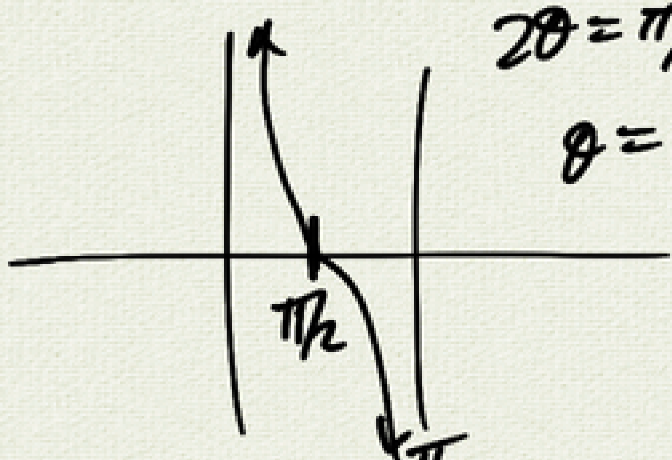
$xy = 1$

$y = \frac{1}{x}$

$xy = 2$

$y = \frac{2}{x}$

$$\cot 2\theta = \frac{A-C}{B} = 0$$



$2\theta = \pi/2$

$\theta = \pi/4$

$$R_\theta = \begin{pmatrix} \sqrt{2}/2 & -\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix} x'-y' \\ x'+y' \end{pmatrix}$$

$x = \sqrt{2}/2\,(x'-y')$

$y = \sqrt{2}/2\,(x'+y')$

$$f(x,y) = xy$$

$$= \frac{\sqrt{2}}{2}(x'-y') \cdot \frac{\sqrt{2}}{2}(x'+y')$$

$$= \frac{1}{2}(x'-y')(x'+y')$$

$$= \frac{1}{2}(x'^2 - y'^2)$$ pringle ($\times \frac{1}{2}$ scale)