4.2 Optimization (Maxima & Minima)

general quadratic

$$Ax^2 + Bxy + Cy^2$$

can remove by rotating coordinate system

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observation:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} Ax + By \\ Bx + Cy \end{pmatrix}$$

$$= Ax^2 + Byx + Bxy + Cy^2$$

$$= Ax^2 + 2Bxy + Cy^2$$

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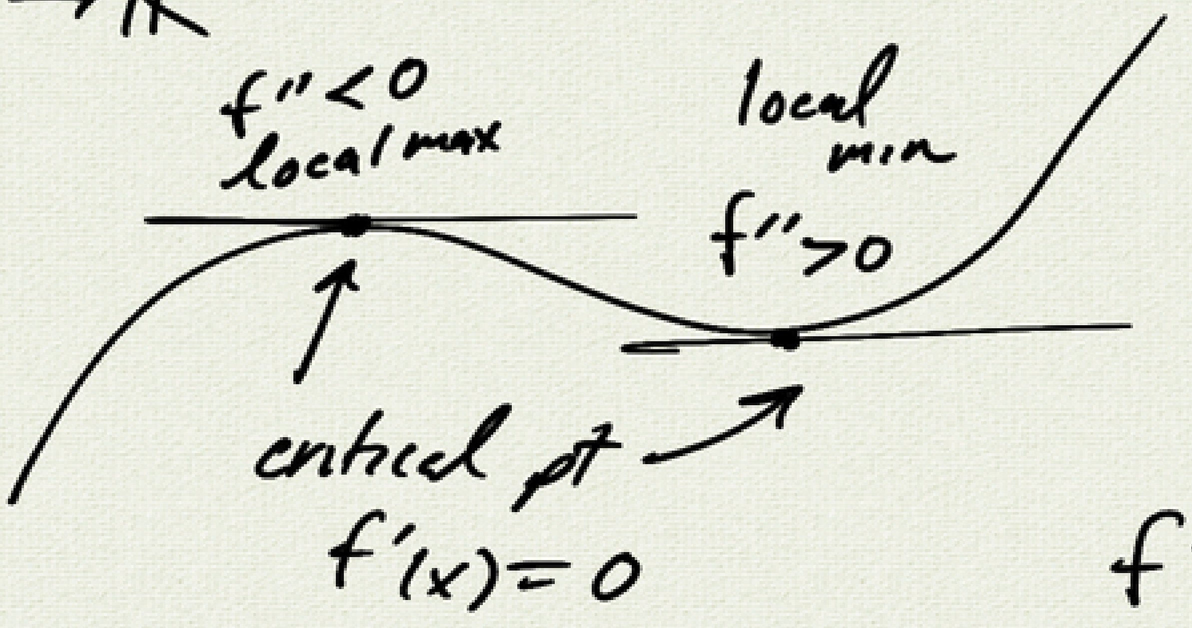
so:

$$Ax^2 + \underline{\underline{B}}xy + Cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2^{nd} derivative test:

$f: \mathbb{R} \to \mathbb{R}$



$f'' = 0 \Rightarrow ?$ we don't know

$f(x) = x^3 \quad g(x) = x^4$

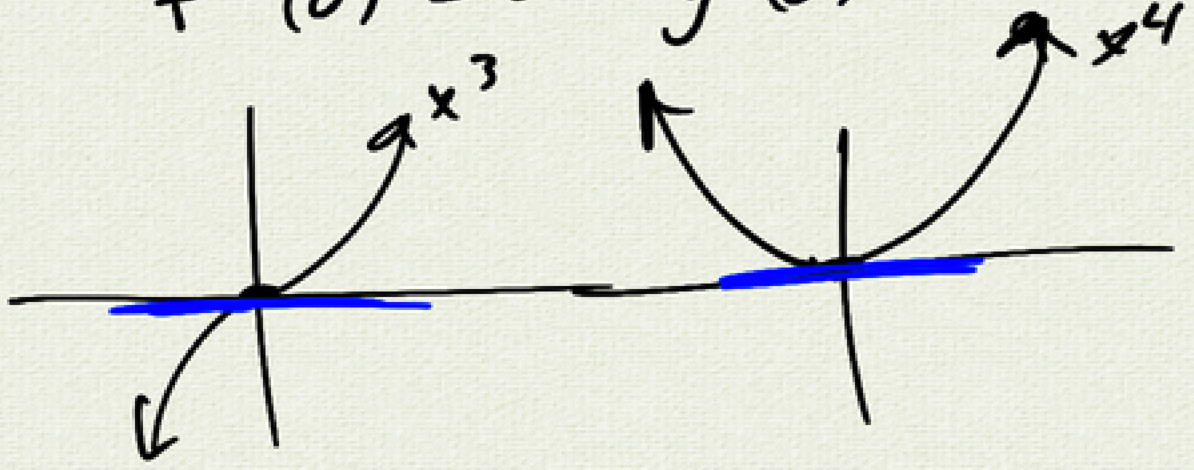
$f'(x) = 3x^2 \qquad g'(x) = 4x^3$

$f''(x) = 6x \qquad g''(x) = 12x^2$

critical pt at $x = 0$

$f'(0) = 0 = g'(0)$

$f''(0) = 0 = g''(0)$



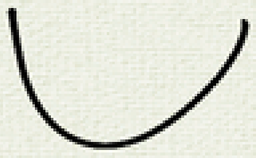
Taylor expansion (2^{nd} order)

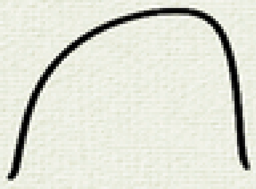
$$f(x) \approx f(x_0) + \underline{f'(x_0)(x - x_0)} + \frac{f''(x_0)(x-x_0)^2}{2}$$

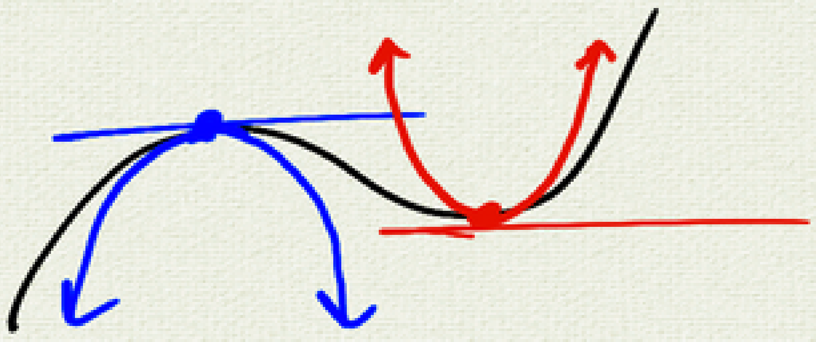
x_0 critical pt $\Rightarrow f'(x_0) = 0$

$$\Rightarrow f(x) \approx f(x_0) + \boxed{\frac{f''(x_0)}{2}}(x - x_0)^2$$

parabola

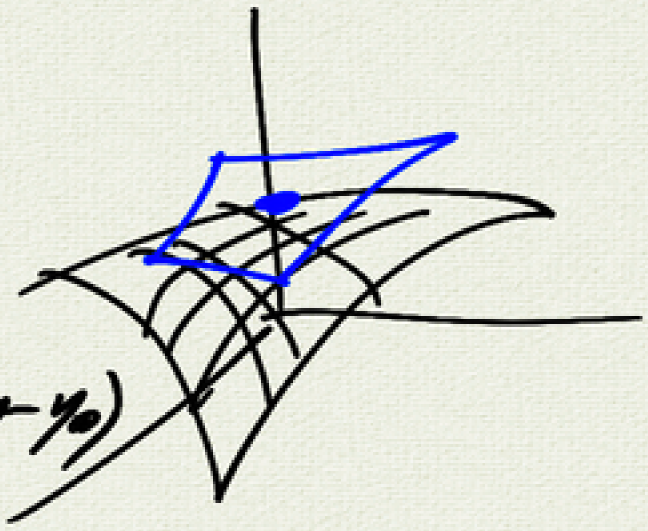
 $f''(x_0) > 0$ local min

 $f''(x_0) < 0$ local max



more dimensions:

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad z = f(x,y)$$
$$(x,y) \mapsto z$$



approximate by tangent plane:

$$f(x,y) \approx f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$$
$$= f(x_0,y_0) + \underbrace{\begin{pmatrix} f_x & f_y \end{pmatrix}}_{df} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \begin{matrix} \Delta x \\ \Delta y \end{matrix}$$

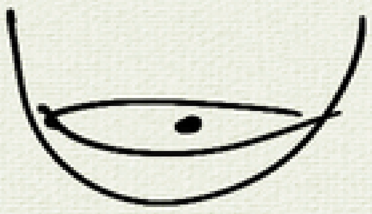
$$f(x,y) \approx f(x_0,y_0) + df \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \underbrace{\tfrac{1}{2} \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} d^2 f \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}}_{\text{2nd order term}}$$

$$\tfrac{1}{2}\left[f_{xx}\Delta x^2 + 2f_{xy}\Delta x \Delta y + f_{yy}\Delta y^2 \right]$$

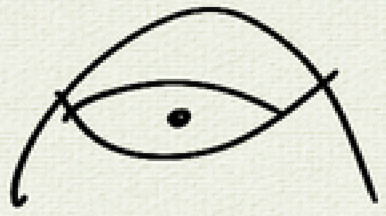
critical pt (x_0,y_0)

$\Rightarrow$ $df(x_0,y_0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$

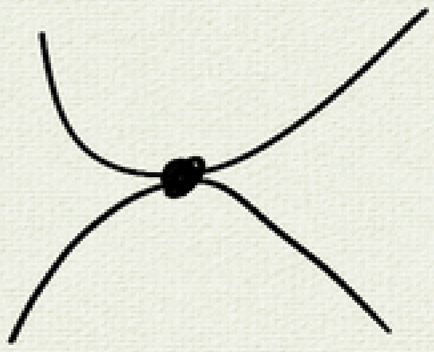
$\Rightarrow$ $f(x,y) \approx$ general quadratic



$\begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$ $\quad \det d^2f > 0$, $f_{xx} > 0$ $\quad$ local min



$\begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}$ $\quad \det d^2f > 0$, $f_{xx} < 0$ $\quad$ local max



$\begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$ $\quad \det d^2f < 0$ $\quad$ saddle pt.

$\begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix}$

$\det d^2f = 0$ $\quad$?

examples

(1) $f(x,y) = xy - 3x - 2y + 6$

$f_x = y - 3$

$f_y = x - 2$

critical pt at $(2,3)$

saddle pt.

$$d^2f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det d^2f = -1$

(2) $g(x,y) = 2x^2 + 2xy + y^2 + 2x + 1$

$g_x = 4x + 2y + 2$

$g_y = 2x + 2y$

$$d^2g = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

critical pts:

$$\begin{array}{r} 4x + 2y + 2 = 0 \\ 2x + 2y = 0 \\ \hline 2x + 2 = 0 \end{array}$$

$x = -1$

$y = 1$

$(-1, 1)$

$\left.\begin{array}{l} \det d^2g = 4 > 0 \\ g_{xx} > 0 \end{array}\right]$ local min

③ $h(x,y) = (4-y^2)\cos x$

$h_x = -(4-y^2)\sin x$ $\qquad d^2h = \begin{pmatrix} -(4-y^2)\cos x & 2y\sin x \\ 2y\sin x & -2\cos x \end{pmatrix}$

$h_y = -2y\cos x$

critical pts: $h_x = 0 \Rightarrow y = \pm 2$ or $\sin x = 0$

$h_y = 0 \rightarrow y = 0$ or $\cos x = 0$

let's look at $(0,0)$:

$$d^2h(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}$$

$\left.\begin{array}{l} \det d^2h = 8 > 0 \\ h_{xx} < 0 \end{array}\right]$ local max from handout:

$$f(x,y) = 2x^2 - 4x + 3y^2 + 12y + 20$$

tangent plane: $z = -4x + 12y + 20$

$$f(x,y) = \underbrace{\left(2x^2 + 3y^2\right)}_{\text{paraboloid at origin}} + \underbrace{\boxed{20 + (-4x) + 12y}}_{\text{plane}}$$