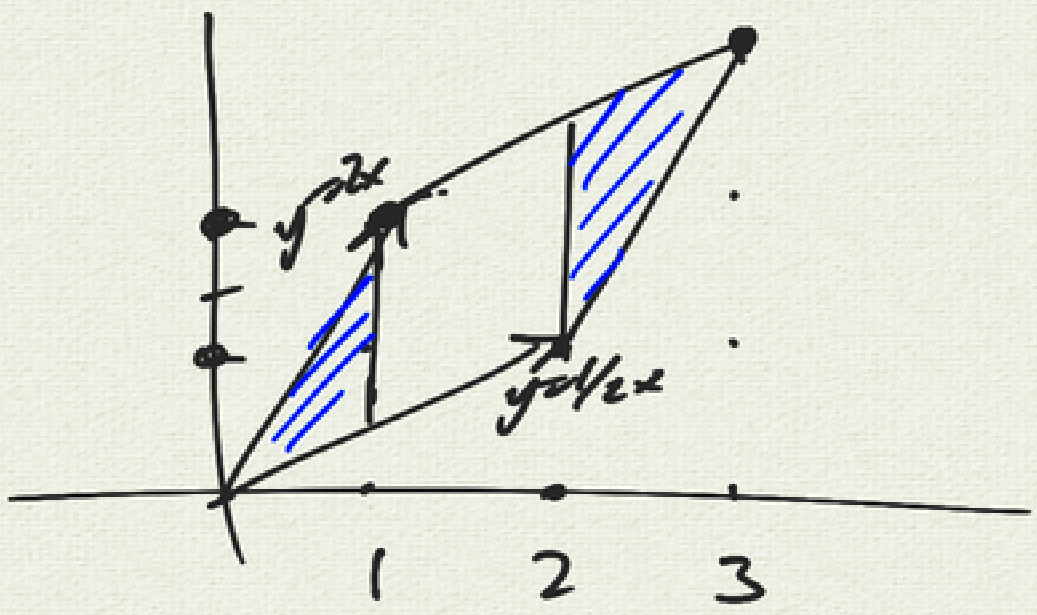
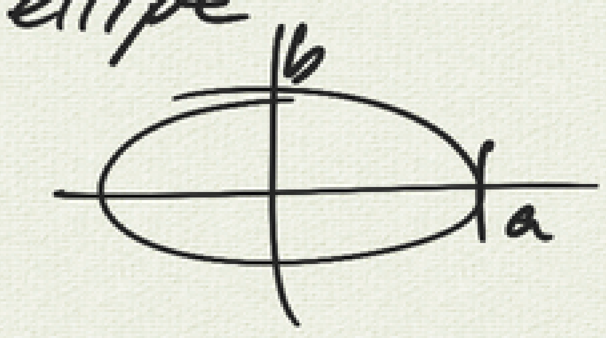
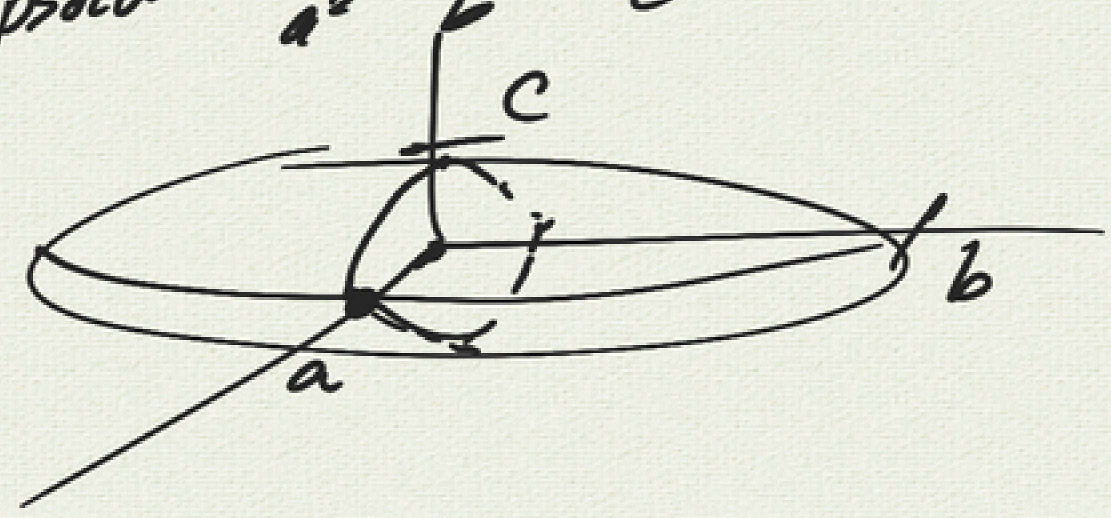


ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

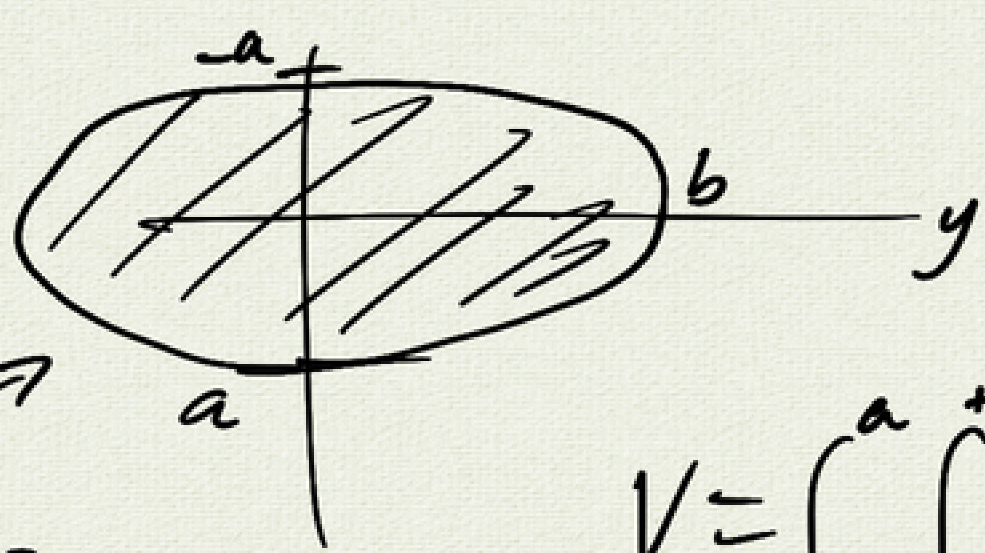


$a=b=c$ sphere

$a=b$ spheroid

prolate (rugby ball) $\updownarrow$

oblate (earth) $\uparrow$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

$$V=\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}}\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=abc$$

$$\boxed{\begin{aligned}x&=au\\ y&=bv\\ z&=cw\end{aligned}}$$

scaling

$u\in[-1,1]$
$v\in[-1,1]$
$w\in[-1,1]$

$$\Rightarrow V=\iiint_{u^2+v^2+w^2\le 1}\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw$$

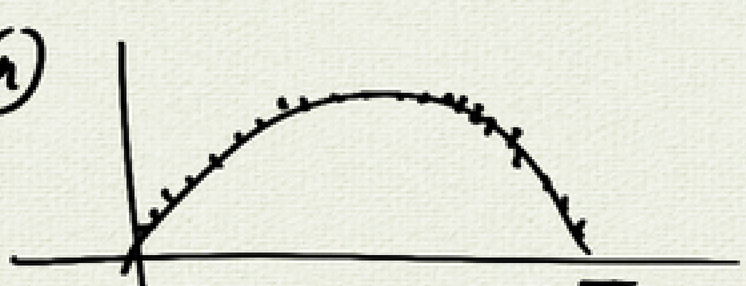
$$=\iiint_{u^2+v^2+w^2\le 1} abc\;du\,dv\,dw$$

$$=abc\,(V_{\text{unit sphere}}) \leftarrow \iiint_{u^2+v^2+w^2\le 1} 1\;du\,dv\,dw$$

$$\boxed{V_{\text{sphere}}=\frac{4}{3}\pi r^3} \qquad \boxed{V_{\text{ellipsoid}}=\frac{4}{3}\pi abc}$$

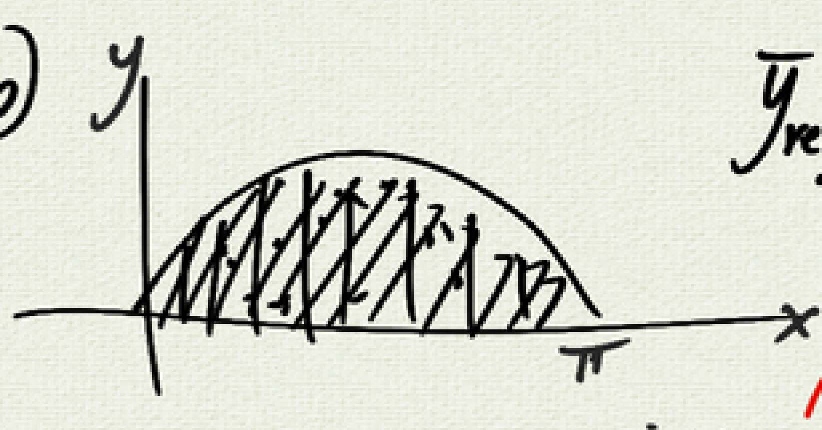
---

3 (a)



$$\bar{y}=\frac{\int \sin x}{\pi}$$

(b)



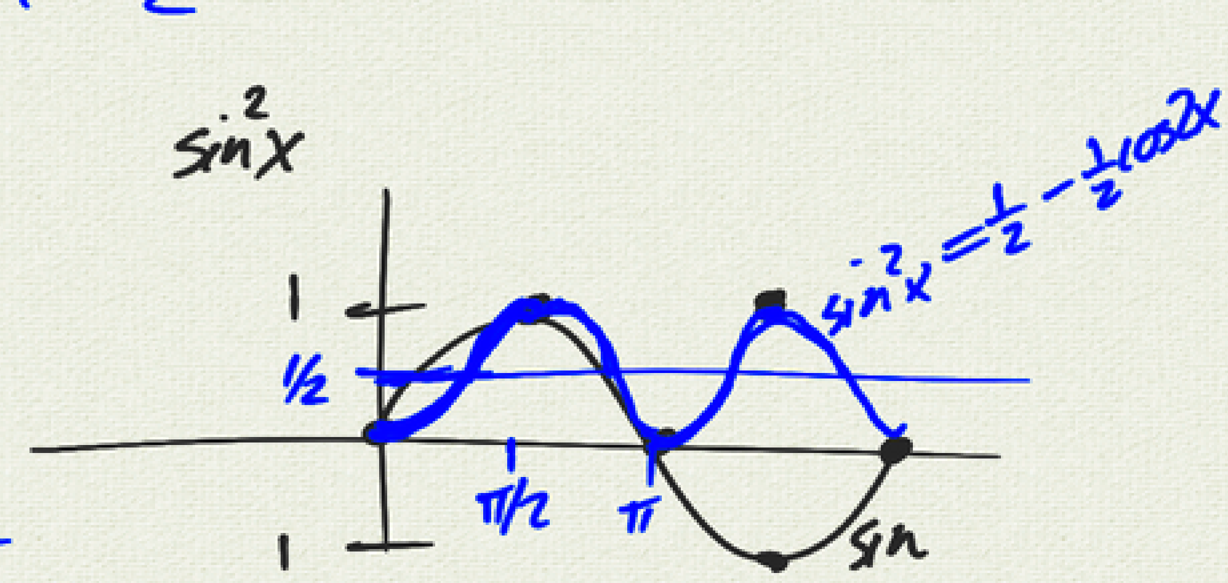
$$\bar{y}_{\text{region}}=\frac{\iint y\,dA}{\iint dA} \leftarrow A=2$$

$$\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx=\int_0^{\pi}\frac{\sin^2 x}{2}dx$$

$$=\frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$=\frac{\pi}{4} \qquad \int_0^{\pi}\cos 2x\,dx=0$$

$\sin^2 x$

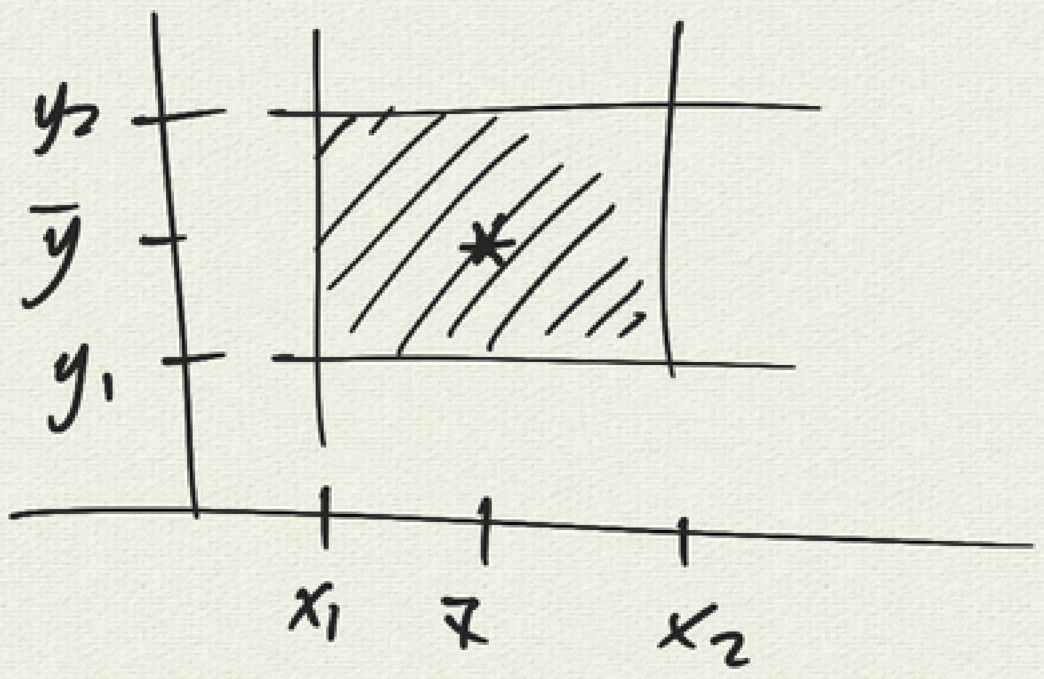


$$\bar{y}_{\text{region}}=\frac{\pi/4}{2}=\pi/8$$

5.10 Centroids

center of mass = centroid

$(\bar{x}, \bar{y})$



should be midpoint

$$\bar{x} = \frac{\iint x \, dy\,dx}{\iint dy\,dx}$$

$$A = \int_{x_1}^{x_2}\int_{y_1}^{y_2} 1 \, dy\,dx$$

$$= (y_2 - y_1)(x_2 - x_1)$$

$$\iint x \, dy\,dx = \int_{x_1}^{x_2}\int_{y_1}^{y_2} x \, dy\,dx$$

$$= \int_{x_1}^{x_2} x(y_2 - y_1)\, dx$$

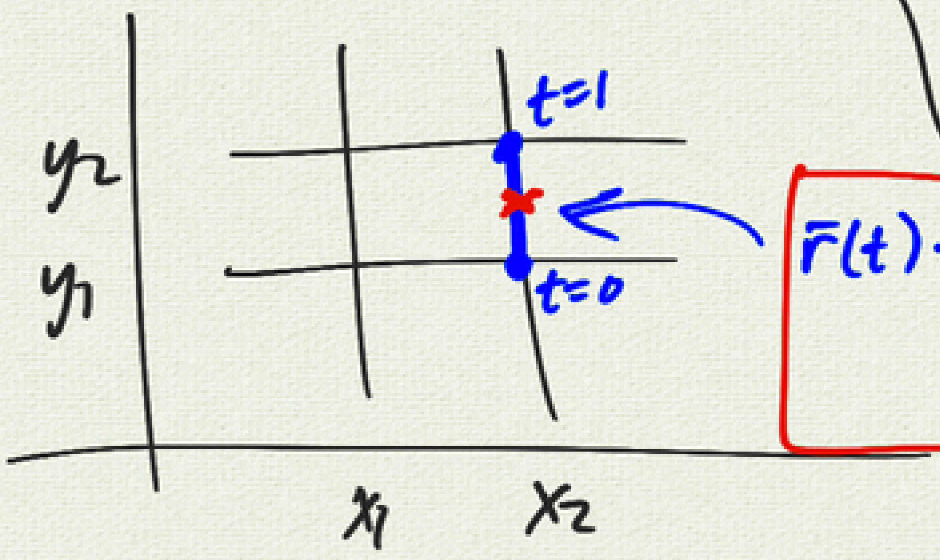
$$= (y_2 - y_1)\int_{x_1}^{x_2} x \, dx$$

$$= (y_2 - y_1)\left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right)$$

$$= \frac{1}{2}(y_2 - y_1)(x_2^2 - x_1^2)$$

$$\bar{x} = \frac{\frac{1}{2}(y_2 - y_1)(x_2^2 - x_1^2)}{(y_2 - y_1)(x_2 - x_1)}$$

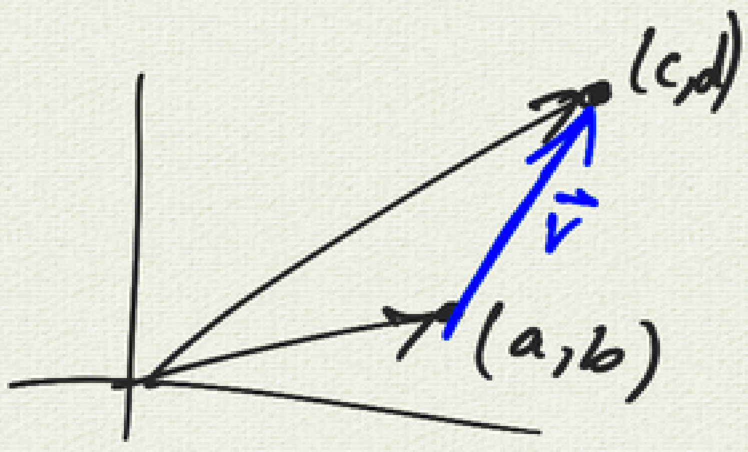
$$= \frac{x_2 + x_1}{2}$$



$$\boxed{\vec{r}(t) = \begin{pmatrix} x_2 \\ y_1 + t(y_2 - y_1) \end{pmatrix}}$$

$$\vec{r}\,'(t) = \begin{pmatrix} 0 \\ y_2 - y_1 \end{pmatrix}$$

$$|\vec{r}\,'(t)| = y_2 - y_1$$



$$\vec{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + t\vec{v}$$

$$\vec{v} = \begin{pmatrix} c-a \\ d-b \end{pmatrix}$$

curve length (arc length)

$$s = \int ds$$

$$= \int |\vec{r}\,'(t)|\, dt$$

$$= \int_0^1 (y_2 - y_1)\, dt$$

$$= y_2 - y_1$$

$$\bar{x} = \frac{\int x \, ds}{\int ds} \leftarrow y_2 - y_1$$

$$ds = |\vec{r}\,'(t)|\, dt$$

$$\int x \, ds = \int x \, |\vec{r}\,'(t)|\, dt$$

$$= \int_0^1 \underbrace{x_2}_{x(t)} (y_2 - y_1)\, dt$$

$$= x_2(y_2 - y_1)$$

$$\Rightarrow \bar{x} = \frac{x_2(y_2 - y_1)}{(y_2 - y_1)} = x_2$$

$$\bar{y} = \frac{\int y \, ds}{\underbrace{\int ds}_{y_2 - y_1}} = \frac{1}{y_2 - y_1}\int y(t)\, |\vec{r}\,'(t)|\, dt$$

$$= \frac{1}{y_2 - y_1}\int_0^1 \big(y_1 + t(y_2 - y_1)\big)(y_2 - y_1)\, dt$$

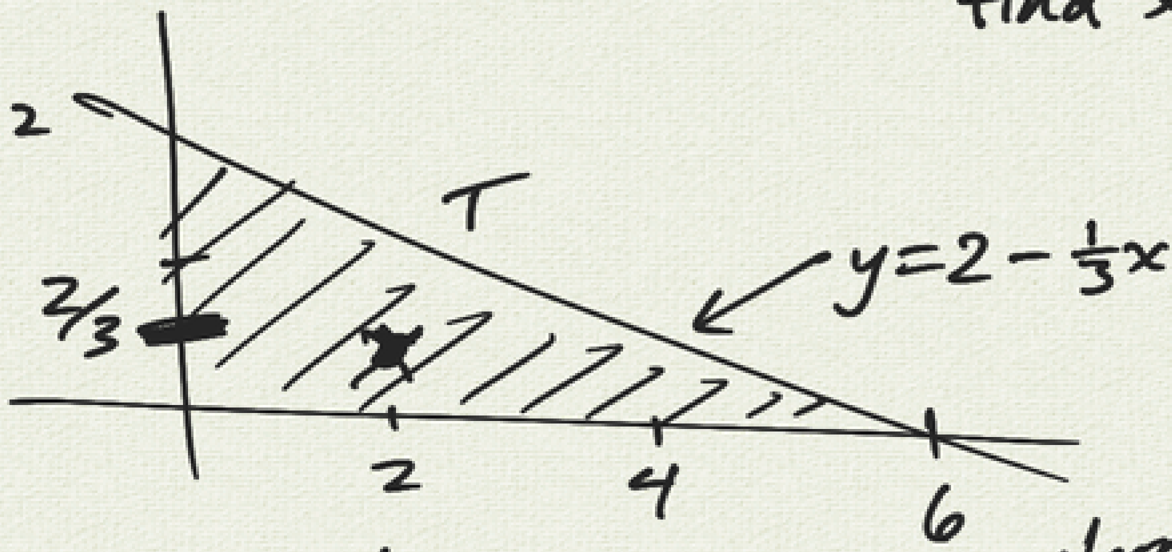
$$= \frac{1}{y_2 - y_1}\left[y_1(y_2 - y_1) + (y_2 - y_1)^2 \frac{1}{2}\right]$$

$$= y_1 + (y_2 - y_1)\frac{1}{2}$$

$$= \frac{y_1 + y_2}{2}$$

we're halfway

triangular region



find $\bar{x} = \dfrac{\iint_T x\, dA}{\iint_T dA}$ ← $A = 6$

check

$$A = \int_0^6 \int_0^{2-\frac{1}{3}x} 1\, dy\, dx$$

$$= \int_0^6 \left(2 - \tfrac{1}{3}x\right) dx$$

$$= 2\cdot 6 - \tfrac{1}{3}\left(\tfrac{6^2}{2}\right)$$

$$= 12 - 6$$

$$= 6$$

top integral:

$$\iint x\, dA = \int_0^6 \int_0^{2-\frac{1}{3}x} x\, dy\, dx$$

$$= \int_0^6 x\left(2 - \tfrac{1}{3}x\right) dx$$

$$= \int_0^6 \left(2x - \tfrac{1}{3}x^2\right) dx$$

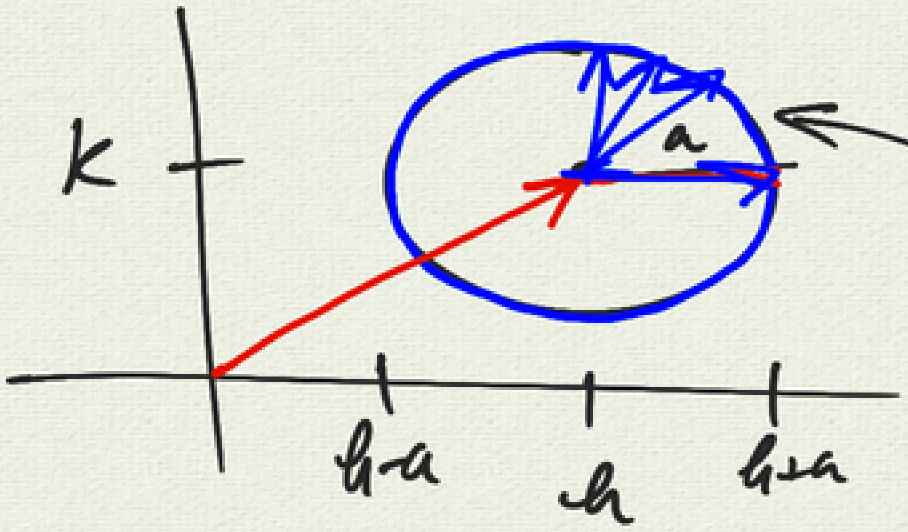
$$= \left[x^2 - \tfrac{1}{9}x^3\right]_0^6$$

$$= 36 - \frac{6^3}{9} \leftarrow \frac{(2\cdot 3)^3}{3^2} = 2^3\cdot 3$$

$$= 36 - 24$$

$$= 12$$

$$\boxed{\bar{x} = 2}$$



circle radius a
center (h,k)

$y=f(x)$

$S=\int ds=\int\sqrt{1+f'(x)^2}\,dx$

$$\boxed{S=\int|\bar{r}'(t)|\,dt}$$

$\bar{r}(x)=\begin{pmatrix}x\\ f(x)\end{pmatrix}$ $\quad\bar{r}'(x)=\begin{pmatrix}1\\ f'(x)\end{pmatrix}$

$(x-h)^2+(y-k)^2=a^2$

$y=k\pm\sqrt{a^2-(x-h)^2}$

$|\bar{r}'(x)|=\sqrt{1+f'(x)^2}$

one parametrization: $\bar{r}(x)=\begin{pmatrix}x\\ k+\sqrt{a^2-(x-h)^2}\end{pmatrix}$

top curve

better parameterization: $\bar{r}(t)=\begin{pmatrix}h\\ k\end{pmatrix}+\begin{pmatrix}a\cos t\\ a\sin t\end{pmatrix}$

$=\begin{pmatrix}h+a\cos t\\ k+a\sin t\end{pmatrix}$ $\leftarrow x(t)$

$\bar{r}'(t)=\begin{pmatrix}-a\sin t\\ a\cos t\end{pmatrix}$ $\quad 0\le t\le 2\pi$

$|\bar{r}'(t)|=a$

$$C=\int ds=\int_0^{2\pi}|\bar{r}'(t)|\,dt=\int_0^{2\pi}a\,dt=2\pi a$$

$$\bar{x}=\frac{\int x\,ds}{\int ds}=\frac{1}{2\pi a}\int x\,|\bar{r}'(t)|\,dt=\frac{1}{2\pi a}\int_0^{2\pi}(h+a\cos t)\,a\,dt=\frac{2\pi ha}{2\pi a}+\frac{a^2}{2\pi a}\underbrace{\int_0^{2\pi}\cos t\,dt}_{0}=h$$