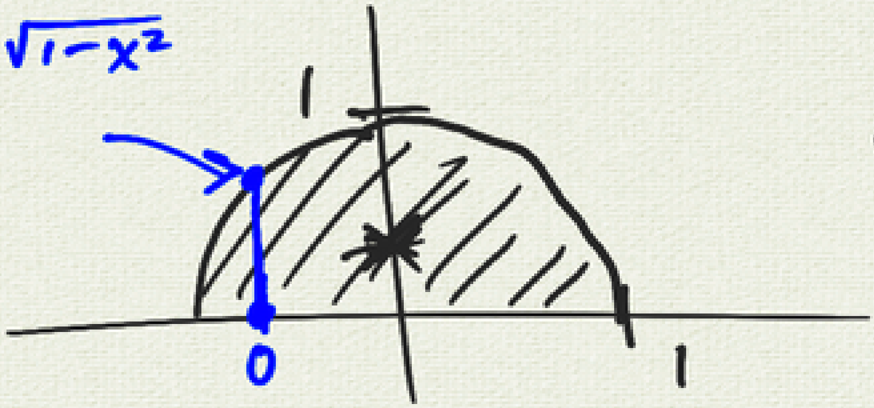
$x^2+y^2=1$

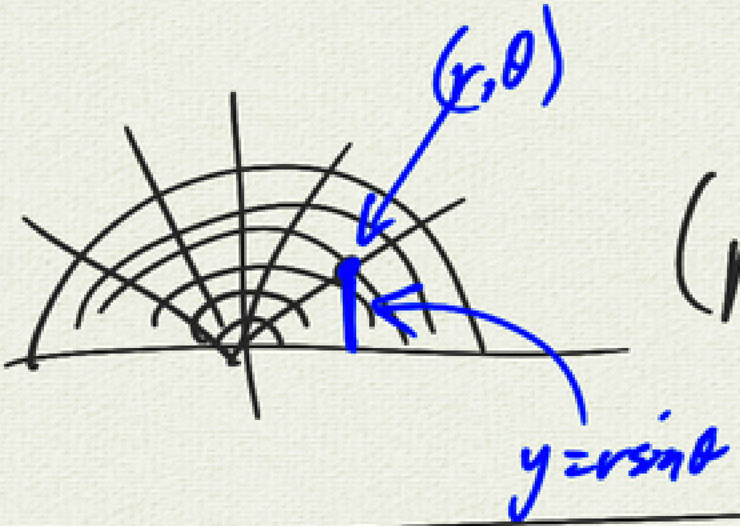
$y=\pm\sqrt{1-x^2}$



$$\bar{y}=\frac{\iint y\,dA}{\iint dA}$$

$\frac{\pi}{2}$

$$=\frac{2}{\pi}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\,dy\,dx$$



(polar)

$$=\frac{2}{\pi}\int_{0}^{\pi}\int_{0}^{1}\underbrace{r\sin\theta}_{y}\,\underbrace{r\,dr\,d\theta}_{dA}$$

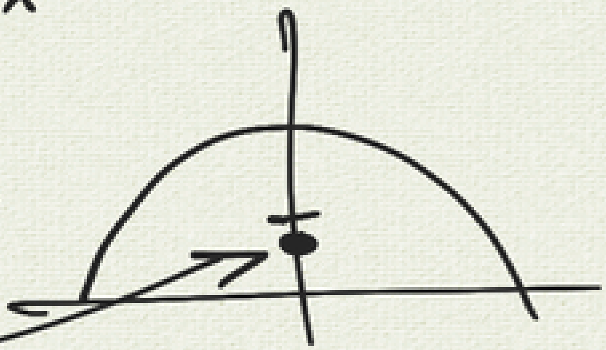
rect

$$=\frac{2}{\pi}\int_{-1}^{1}\frac{(\sqrt{1-x^2})^2}{2}\,dx$$

$$=\frac{1}{\pi}\int_{-1}^{1}(1-x^2)\,dx$$

$$=\frac{1}{\pi}\left[x-\frac{x^3}{3}\right]_{-1}^{1}$$

$$=\frac{4}{3\pi}$$



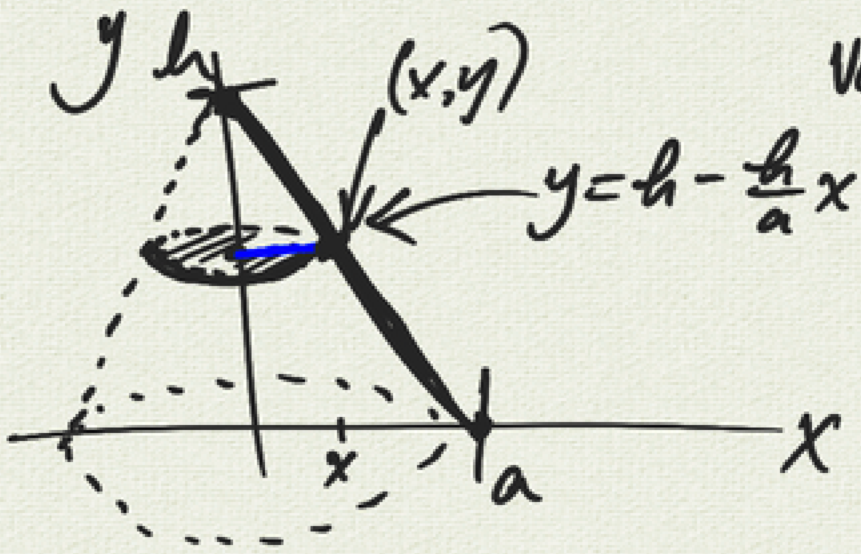
polar

$$=\frac{2}{\pi}\int_{0}^{\pi}\sin\theta\left(\frac{1}{3}\right)d\theta$$

$$=\frac{2}{3\pi}\underbrace{\int_{0}^{\pi}\sin\theta\,d\theta}_{2}$$

$$=\frac{4}{3\pi}$$

5.11 Disks and shells



Volume of revolution

disks:

$$V = \int_0^h (\pi x^2)\,dy$$

$$= \int_0^h \pi \underbrace{\left(\frac{a}{h}\right)^2 (y-h)^2}_{x^2} dy$$

$$= \frac{\pi a^2}{h^2}\int_0^h (y-h)^2\,dy$$

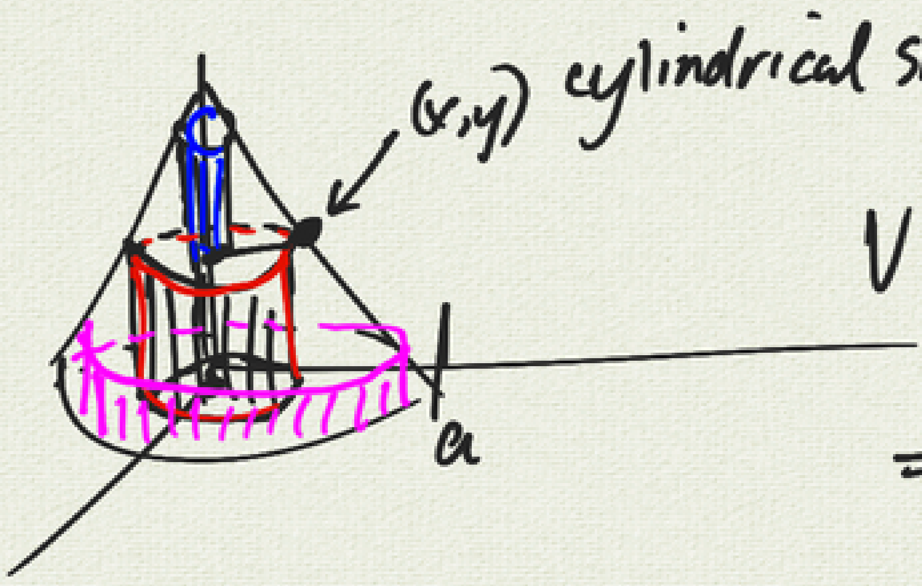
$$= \frac{\pi a^2}{h^2} \left.\frac{(y-h)^3}{3}\right|_0^h$$

$$= \frac{\pi a^2 h}{3}$$

$$y = h - \frac{h}{a}x$$

$$y - h = -\frac{h}{a}x$$

$$x = -\frac{a}{h}(y-h)$$



cylindrical shells

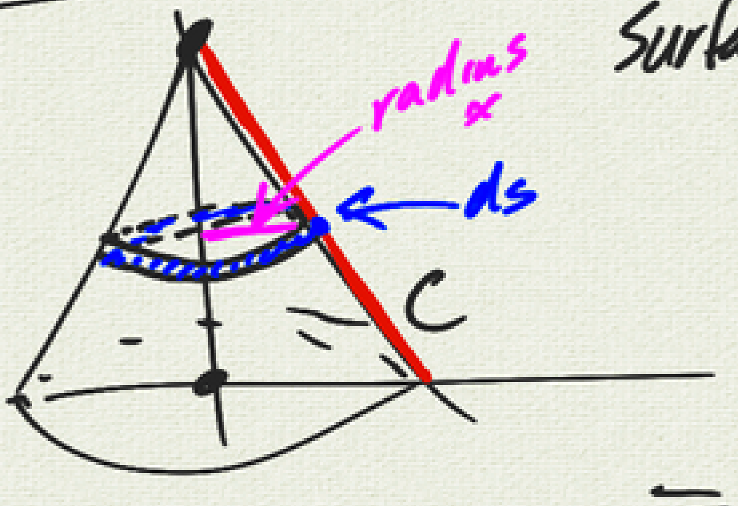
$$V = \int_0^a 2\pi \overset{\text{radius}}{x}\ \overset{\text{height}}{y}\ \ dx$$

$$= \boxed{2\pi\int_0^a x\left(h - \frac{h}{a}x\right)dx}$$

$$= 2\pi\left[\frac{hx^2}{2} - \frac{h}{a}\frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[\frac{ha^2}{2} - \frac{ha^2}{3}\right]$$

$$= \frac{\pi h a^2}{3}$$



Surface area

$$SA = \int_C 2\pi x\,ds$$

$$= \int_0^1 2\pi \underset{x(t)}{x}\ \underbrace{|\bar r'(t)|\,dt}_{ds}$$

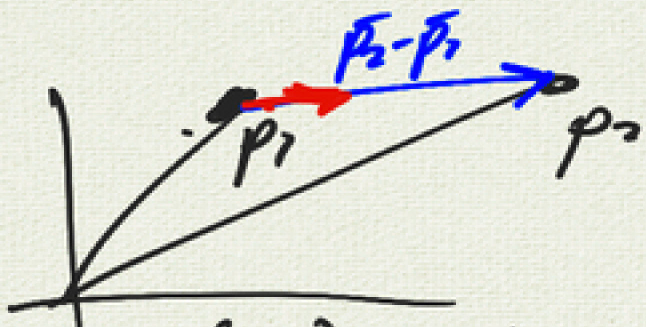
$$= 2\pi\int_0^1 \underbrace{at}_{x}\ \underbrace{l}_{|\bar r'(t)|}\ dt$$

$$= \pi a l$$

parametrization

$$\underset{\bar p_1}{\begin{pmatrix}0\\h\end{pmatrix}} \to \underset{\bar p_2}{\begin{pmatrix}a\\0\end{pmatrix}}$$

$$\bar r(t) = \bar p_1 + t(\bar p_2 - \bar p_1)$$

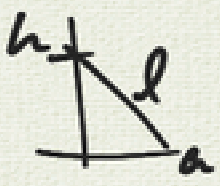


$$= (1-t)\bar p_1 + t\bar p_2$$

$$\bar r(t) = \begin{pmatrix}at\\ h(1-t)\end{pmatrix}$$

$$\bar r'(t) = \begin{pmatrix}a\\ -h\end{pmatrix} \quad |\bar r'(t)| = \sqrt{a^2+h^2}$$

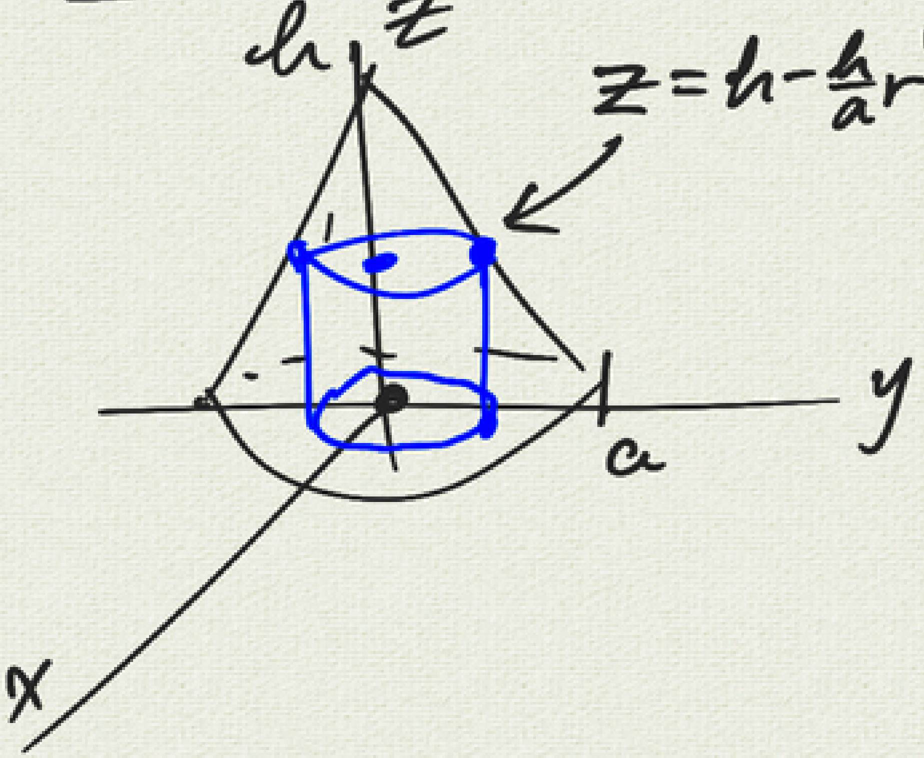
$$= l$$



slant height

cone (in 3D)

cylindrical : (r, θ, z)



$$V = \iiint dV$$

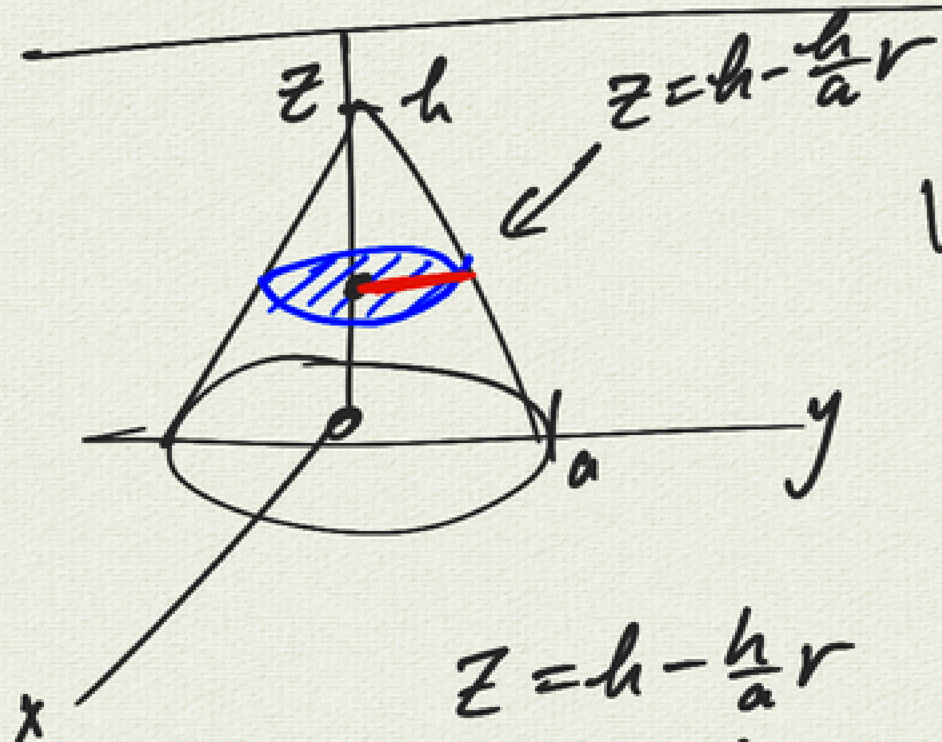
$$= \int_0^a \int_0^{2\pi} \left[\int_0^{h - \frac{h}{a} r} r \, dz \right] d\theta \, dr$$

$$= 2\pi \int_0^a \int_0^{h - \frac{h}{a} r} r \, dz \, dr$$

$$= 2\pi \int_0^a r \left(h - \frac{h}{a} r\right) dr$$

cylindrical shells $\Longrightarrow$

$$= \int_0^a (2\pi r) \left(h - \frac{h}{a} r\right) dr$$



$$V = \int_0^h \int_0^{2\pi} \int_0^{-\frac{a}{h}(z-h)} r \, dr \, d\theta \, dz$$

$$= 2\pi \int_0^h \left[\frac{r^2}{2} \right]_0^{-\frac{a}{h}(z-h)} dz$$

$z = h - \frac{h}{a} r$

$z - h = -\frac{h}{a} r$

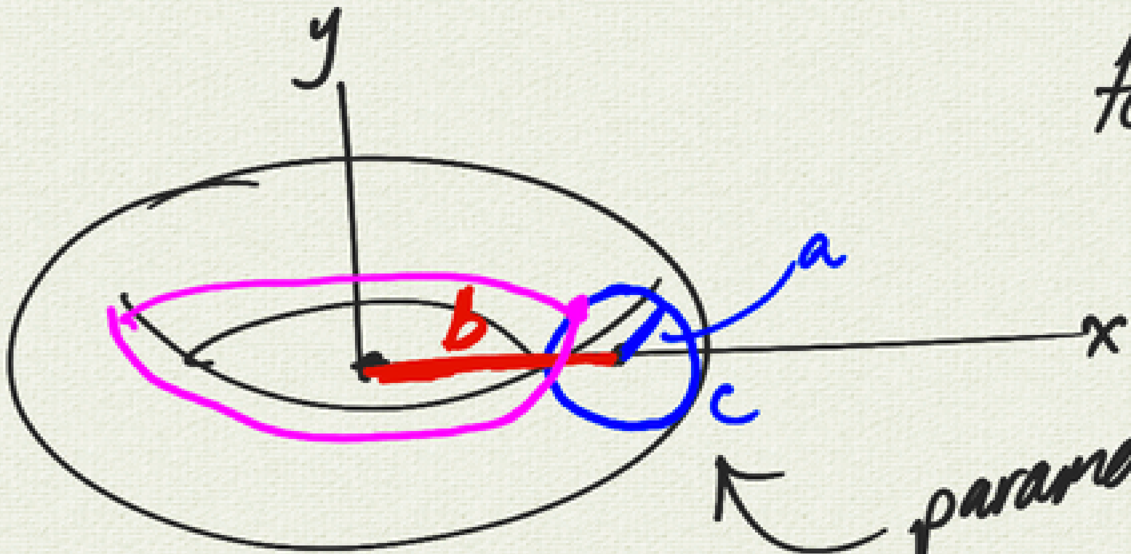
$r = -\frac{a}{h}(z - h)$

$$= 2\pi \int_0^h \frac{1}{2} \left(\frac{a}{h}\right)^2 (z-h)^2 \, dz$$

$$= \int_0^h \pi \underbrace{\left(\frac{a}{h}\right)^2 (z-h)^2}_{\text{radius}^2} dz$$

disks $\Longrightarrow$

torus



parametrize circle

$$\bar r(t) = \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix} = \begin{pmatrix} b + a\cos t \\ a\sin t \end{pmatrix}$$

$$\bar r'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix} = a\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$$|\bar r'(t)| = a$$

$$SA = \int_C 2\pi x \, ds$$

$$= \int_0^{2\pi} 2\pi \underbrace{(b + a\cos t)}_{x} \, \underbrace{|\bar r'(t)| \, dt}_{ds}$$

$$= 2\pi \int_0^{2\pi} (b + a\cos t) a \, dt$$

$$= (2\pi a)(2\pi b) + 2\pi a^2 \underbrace{\int_0^{2\pi} \cos t \, dt}_{0}$$

$$2\pi ab \underbrace{\boxed{\int_0^{2\pi} dt}}_{2\pi}$$

$$SA = (2\pi a)(2\pi b)$$