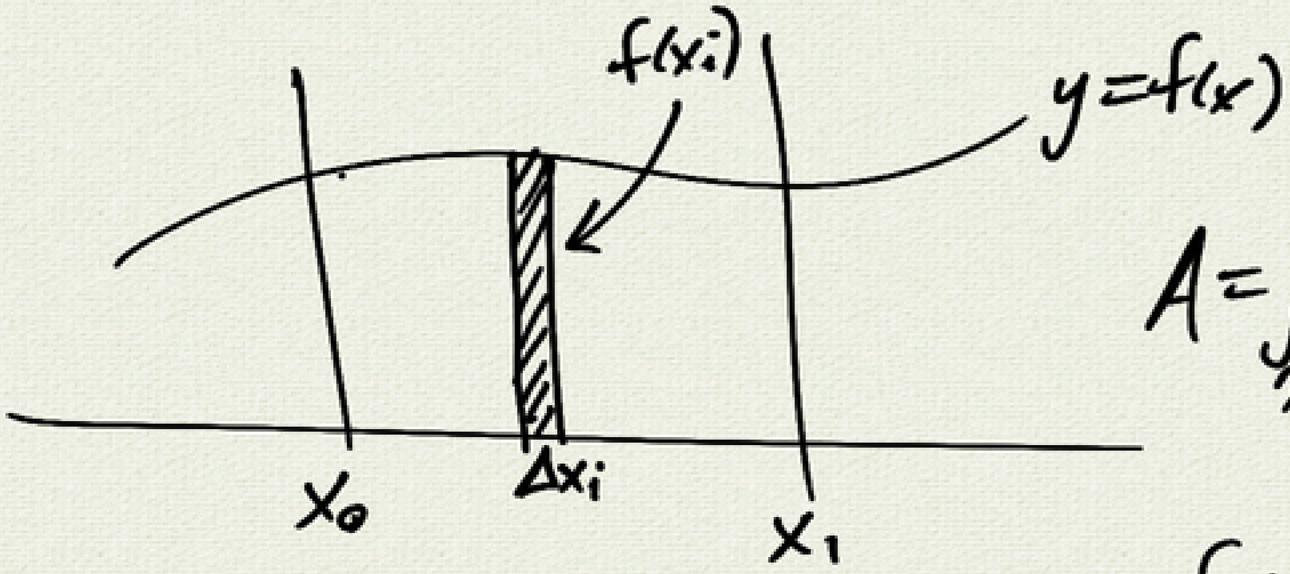
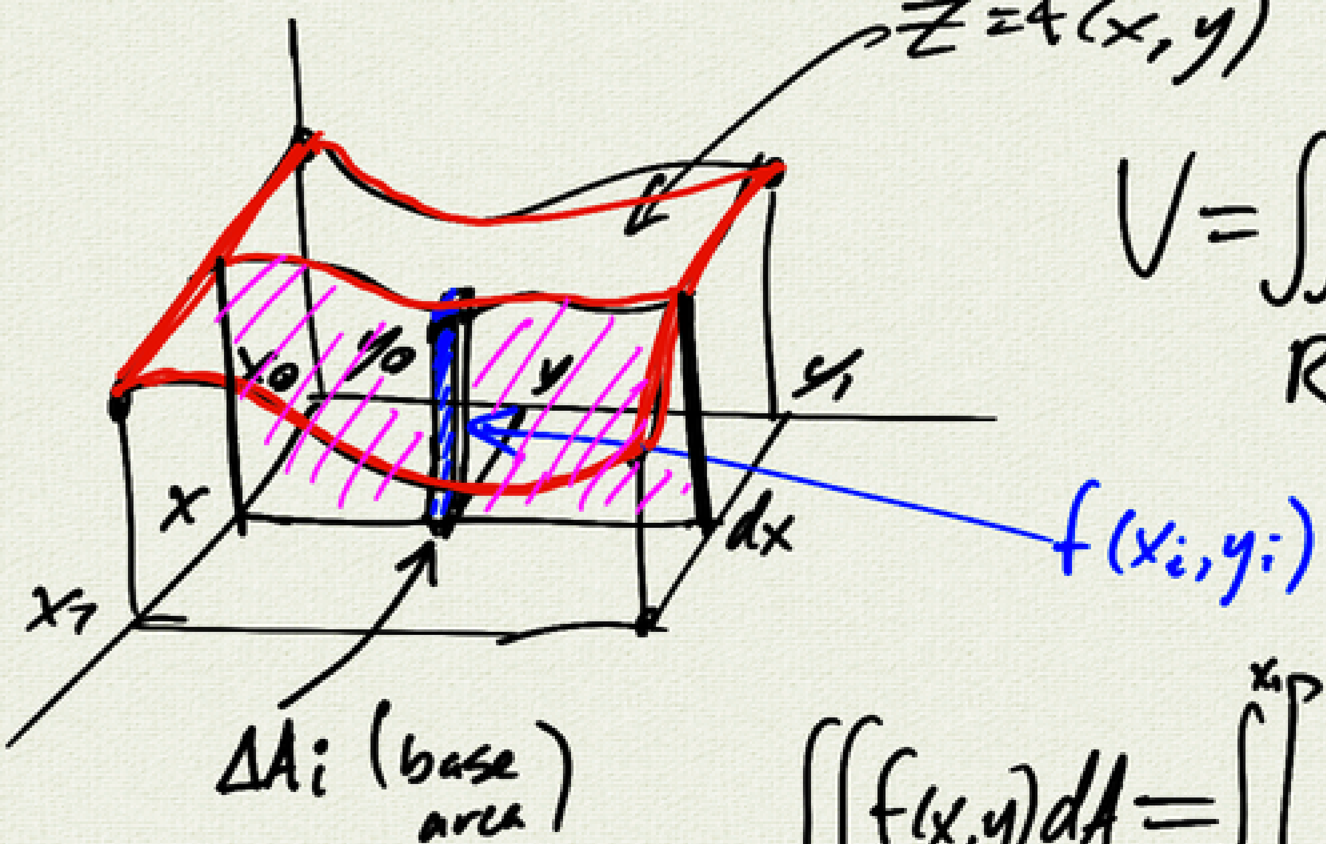
5.1 Double Integrals



$$A=\int_{x_0}^{x_1} f(x)\,dx = \lim \sum_i f(x_i)\,\Delta x_i$$

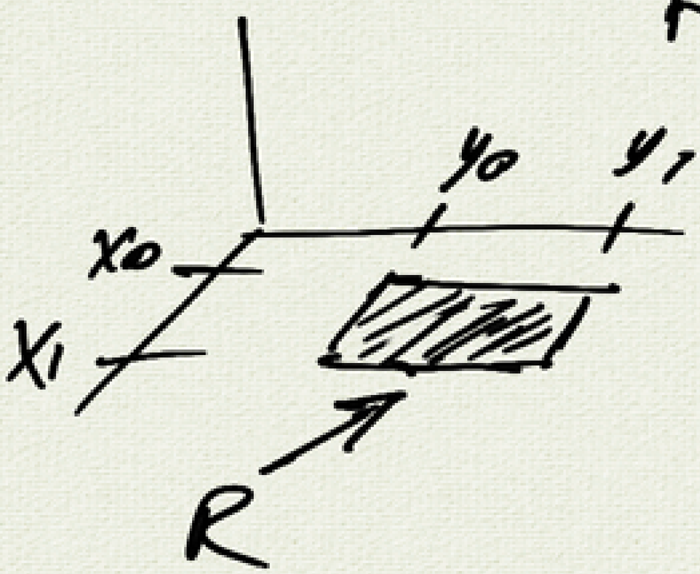


$$V=\iint_R f(x,y)\,dA = \lim \sum_i f(x_i,y_i)\,\Delta A_i$$

$$\iint_R f(x,y)\,dA = \int_{x_0}^{x_1}\left[\int_{y_0}^{y_1} f(x,y)\,dy\right]dx$$

$$= \int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} f(x,y)\,dx\right]dy$$

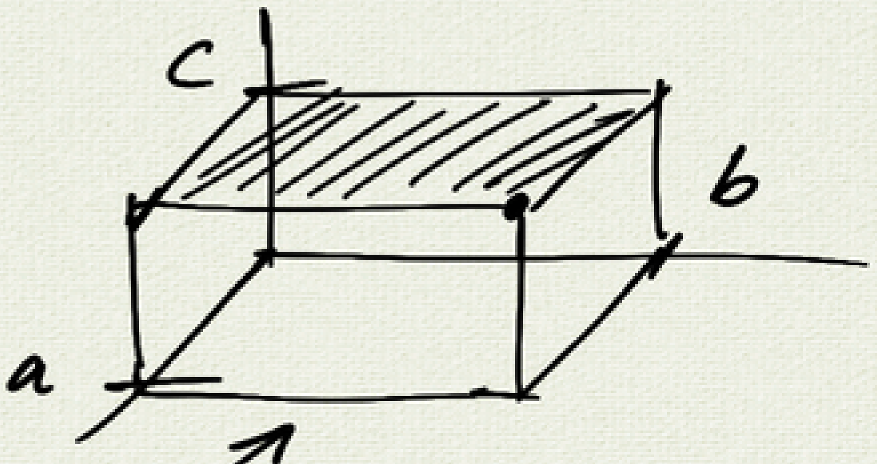
$\underbrace{[x_0,x_1]\times[y_0,y_1]}_{\text{rectangle}} \to R$



$$R=\{(x,y)\mid x\in[x_0,x_1] \text{ and } y\in[y_0,y_1]\}$$

example:

$$\boxed{f(x,y) = c}$$



$R = [0,a]\times[0,b]$

$$= \iint_R f(x,y)\,dA$$

$$V = \int_0^a \left[\int_0^b f(x,y)\,dy\right] dx$$

$y \in [0,b]$

$$= \int_0^a \left[\int_0^b c\,dy\right] dx$$

$$= \int_0^a [cy]_0^b\,dx$$

$$= \int_0^a cb\,dx$$

$$V = abc$$

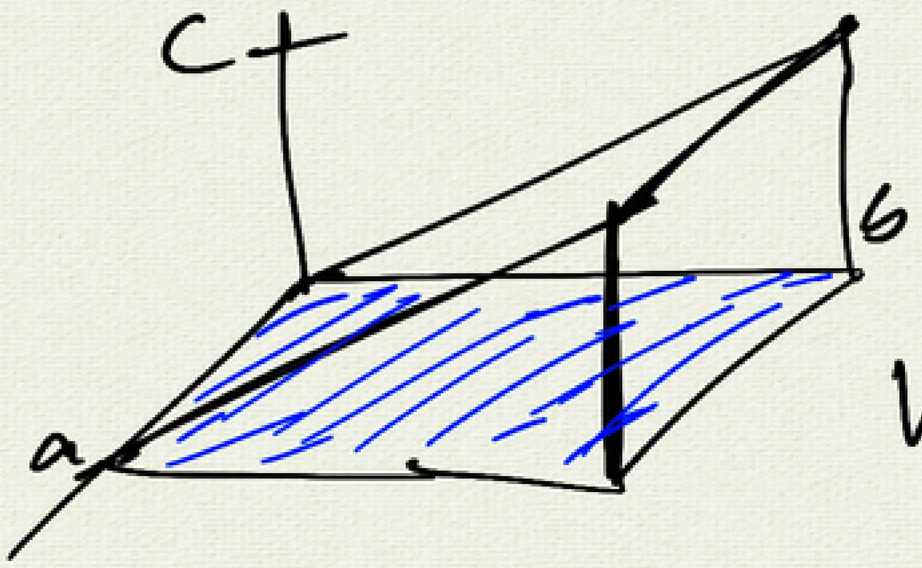
if $F'(x) = f(x)$,

then

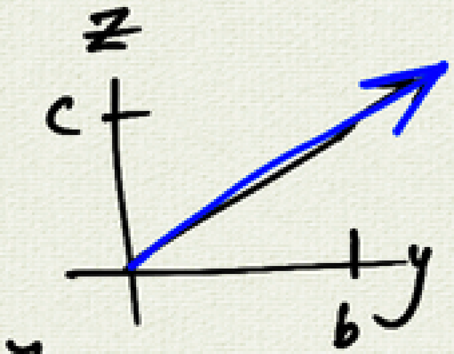
$$\int_0^b f(x)\,dx = F(b) - F(a)$$

Fundamental Theorem of Calculus

example:



$$f(x,y) = \frac{c}{b}y$$



$R = [0,a]\times[0,b]$

$$z = \frac{c}{b}y$$

$$z = ax + by$$
$$= f_x x + f_y y$$

$$V = \iint_R f(x,y)\,dA$$

$$= \int_0^a \left[\int_0^b \frac{c}{b}y\,dy\right] dx$$

$$= \int_0^a \frac{c}{b}\left[\frac{y^2}{2}\right]_0^b dx$$

$$= \int_0^a \frac{bc}{2}\,dx$$

$$= \frac{abc}{2}$$

example

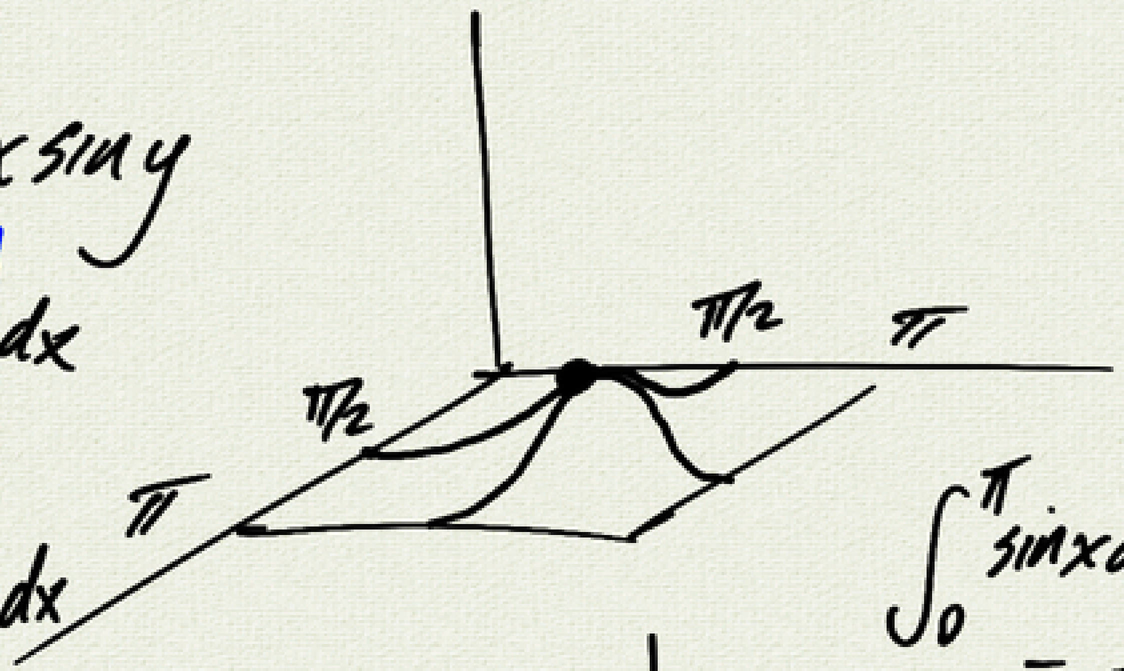
$$f(x,y) = \sin x \sin y$$

$$V = \int_0^{\pi}\left[\int_0^{\pi} \sin x \sin y \, dy\right] dx$$

$$= \int_0^{\pi} \sin x \left[\int_0^{\pi} \sin y \, dy\right] dx$$

$$= 2\int_0^{\pi} \sin x \, dx$$

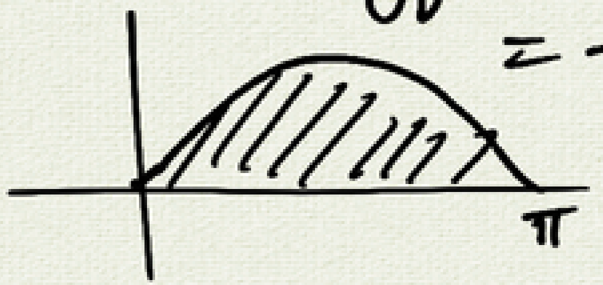
$$= 4$$



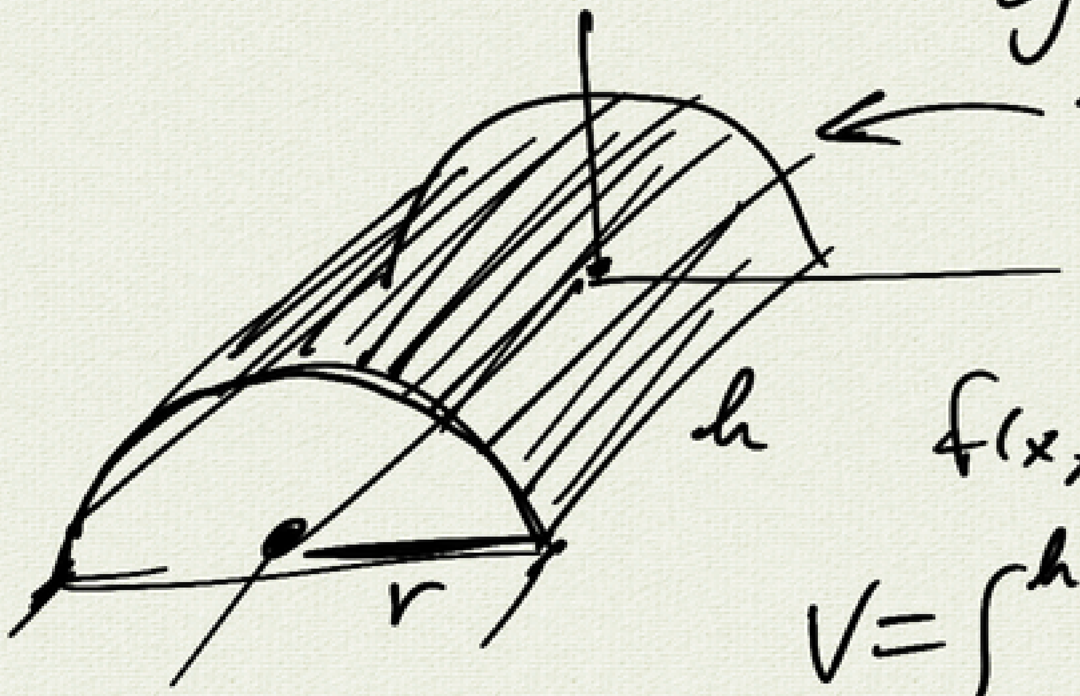
$$\int_0^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_0^{\pi}$$

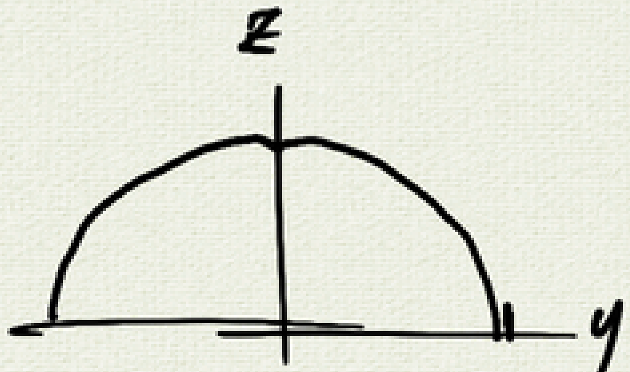
$$= 2$$



example



cylinder $f(x,y) = z$



$y^2+z^2=r^2$

$z=\sqrt{r^2-y^2}$

$f(x,y)=\sqrt{r^2-y^2}$

$$V=\int_0^h\int_{-r}^{r}\sqrt{r^2-y^2}\,dy\,dx$$

$$=\int_{-r}^{r}\left[\int_0^h\sqrt{r^2-y^2}\,dx\right]dy$$

$$=h\int_{-r}^{r}\sqrt{r^2-y^2}\,dy$$

$$=h\int_{-\pi/2}^{\pi/2}\underbrace{\sqrt{r^2-r^2\sin^2\theta}}_{r\cos\theta}\cdot(r\cos\theta\,d\theta)$$

$$=hr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

trig substitution

$\cos^2\theta+\sin^2\theta=1$

$\boxed{y=r\sin\theta}$

$r^2-y^2=\underbrace{r^2-r^2\sin^2\theta}_{r^2\cos^2\theta}$

$\boxed{dy=r\cos\theta\,d\theta}$

$\cos 2\theta=\cos^2\theta-\sin^2\theta$

$=1-2\sin^2\theta$

$=2\cos^2\theta-1$

$\Rightarrow \cos^2\theta=\dfrac{1+\cos 2\theta}{2}$ power reducing

$$=hr^2\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta\right]d\theta$$

check $\int_{-\pi/2}^{\pi/2}\cos 2\theta\,d\theta=0$

$$=\frac{\pi h r^2}{2}$$