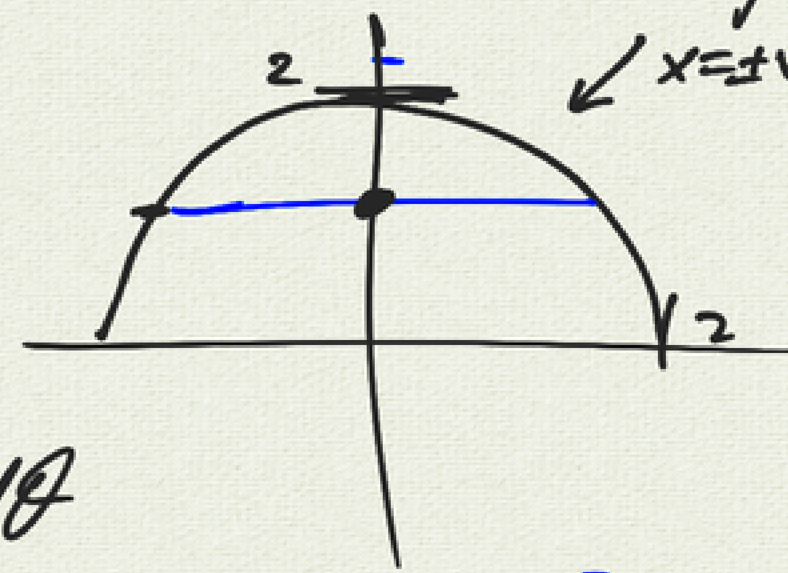
149

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \underbrace{\boxed{(x^2+y^2)^2}} dx\,dy$$

$x^2+y^2=4$

$x=\pm\sqrt{4-y^2}$

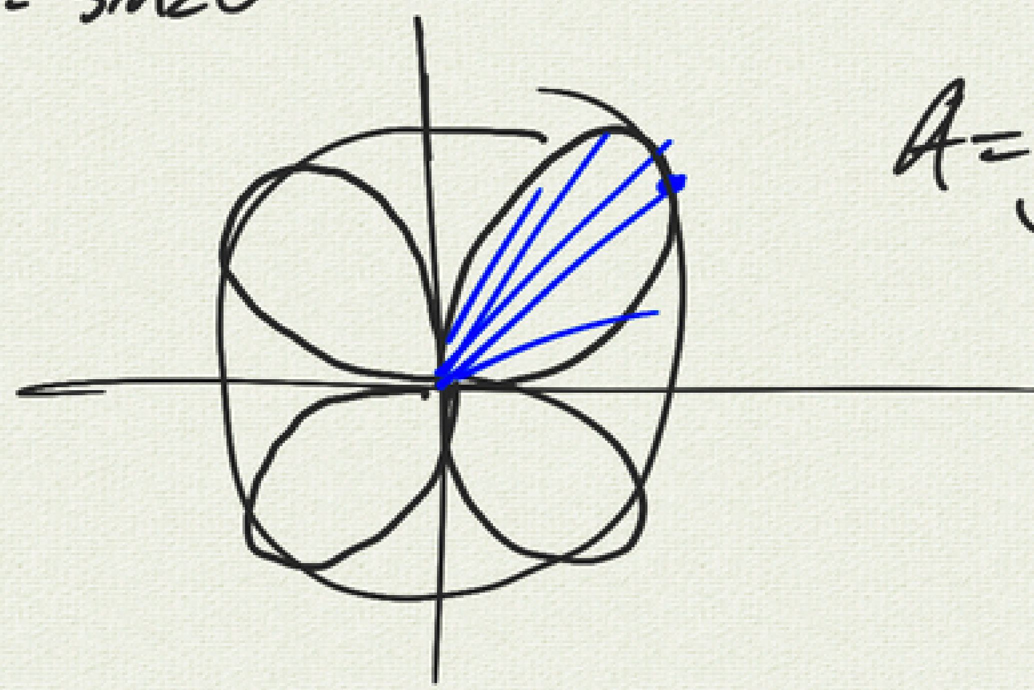


$$= \int_0^{\pi} \int_0^2 (\underbrace{r^2})^2 \, r\,dr\,d\theta$$

$x^2+y^2=r^2$

---

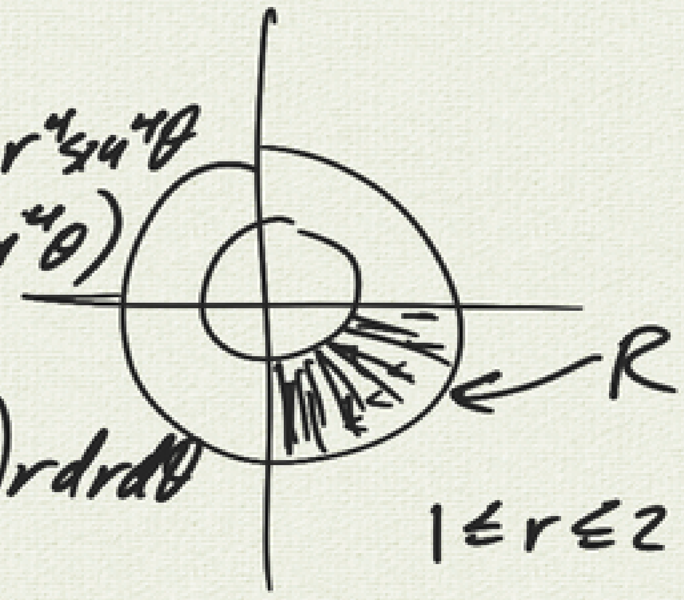
155 $r = \sin 2\theta$



$$A = \int_0^{2\pi} \int_0^{\sin 2\theta} \underbrace{r\,dr\,d\theta}_{dA}$$

$= \pi/2$

(137) $f(x,y) = x^4 + y^4 = r^4\cos^4\theta + r^4\sin^4\theta$
$= r^4(\cos^4\theta + \sin^4\theta)$



$1 \le r \le 2$

$\frac{3\pi}{2} \le \theta \le 2\pi$

$$\iint_R f(x,y)\,dA = \int_{3\pi/2}^{2\pi}\int_1^2 r^4(\cos^4\theta + \sin^4\theta)\,r\,dr\,d\theta$$

$$= \int_{3\pi/2}^{2\pi} (\cos^4\theta + \sin^4\theta)\left[\int_1^2 r^5\,dr\right] d\theta$$

$$\left.\frac{r^6}{6}\right|_1^2 = \frac{1}{6}[2^6 - 1] = \frac{63}{6} = \frac{21}{2}$$

$$= \frac{21}{2}\underbrace{\int_{3\pi/2}^{2\pi} (\cos^4\theta + \sin^4\theta)\,d\theta}$$

$$\int_{3\pi/2}^{2\pi} \cos^4\theta\,d\theta \qquad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$= \int_{3\pi/2}^{2\pi} \frac{1}{4}(1 + \cos 2\theta)^2\,d\theta$$

$$= \frac{1}{4}\int_{3\pi/2}^{2\pi} (1 + \underbrace{2\cos 2\theta}_{0} + \cos^2 2\theta)\,d\theta \Leftarrow \text{check}$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right) + \frac{1}{4}\int_{3\pi/2}^{2\pi} \frac{1}{2}(1 + \underline{\cos 4\theta})\,d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right) + \frac{1}{8}\left(\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{16}$$

also: $\int_{3\pi/2}^{2\pi} \sin^4\theta\,d\theta = \frac{3\pi}{16}$

---

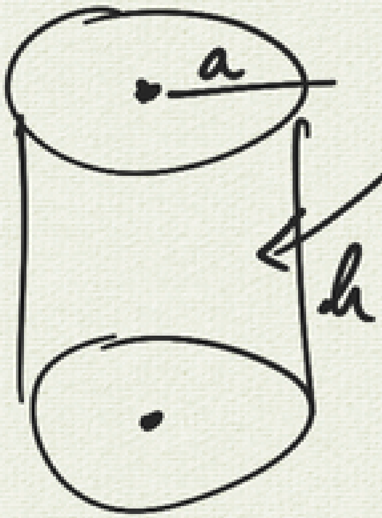
final: $\frac{21}{2}\left(\frac{3\pi}{16} \cdot 2\right) = \frac{63\pi}{16}$

5.5 Parametric surfaces

geometry:

$\boxed{\text{cylinder}}$

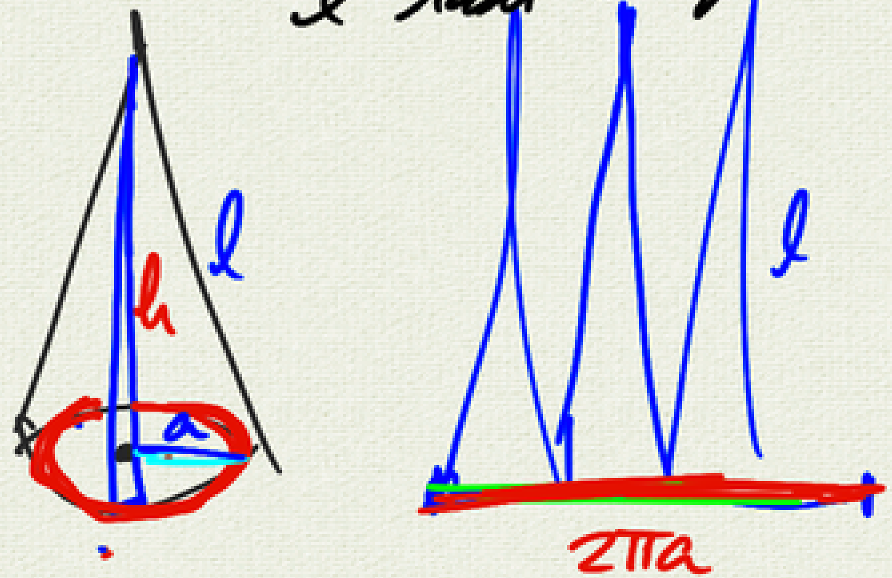
$SA = 2\pi a h$



$\boxed{\text{cone}}$

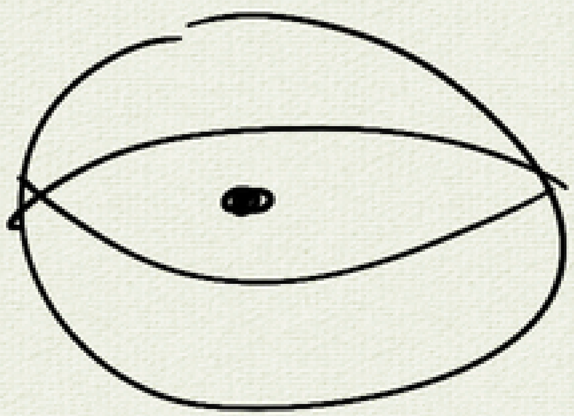
$l^2 = h^2 + a^2$

l slant height



$SA = \frac{1}{2}(2\pi a) l = \pi a l$

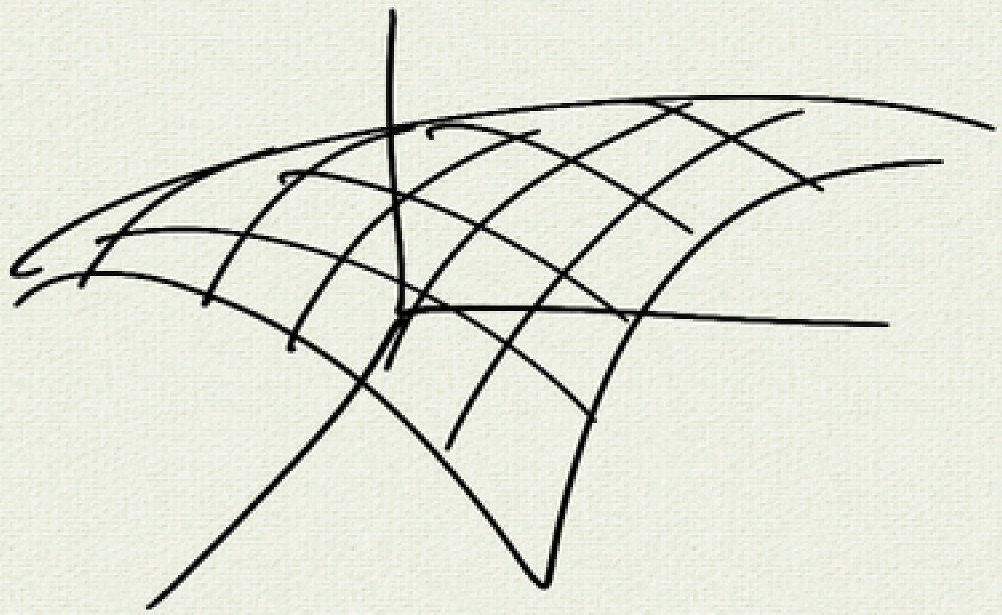
$\boxed{\text{sphere}}$



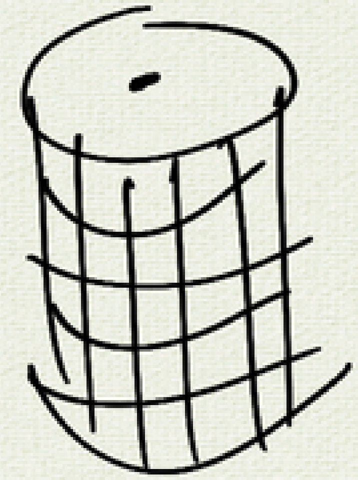
$V = \frac{4}{3}\pi r^3$

$\Rightarrow SA = 4\pi r^2$

surfaces: $z = f(x,y)$



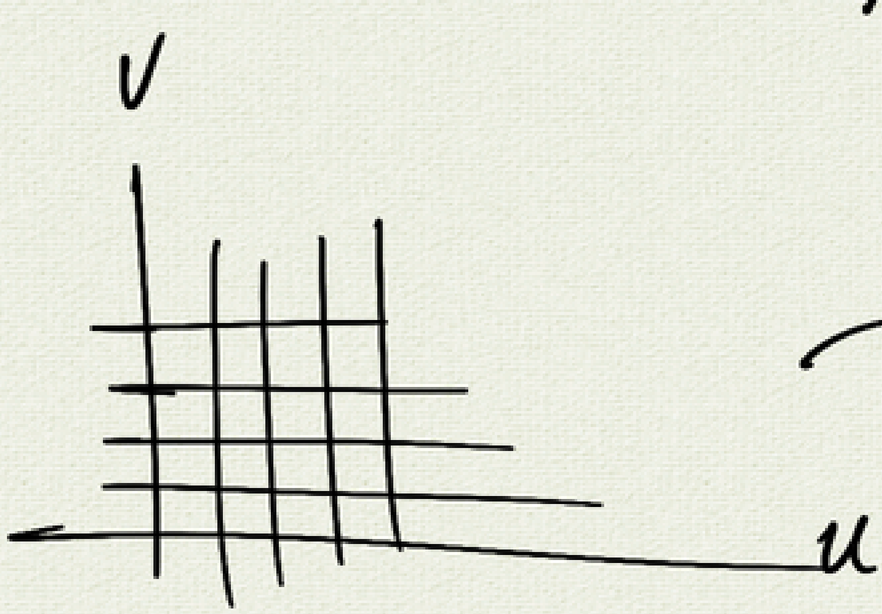
but: ?



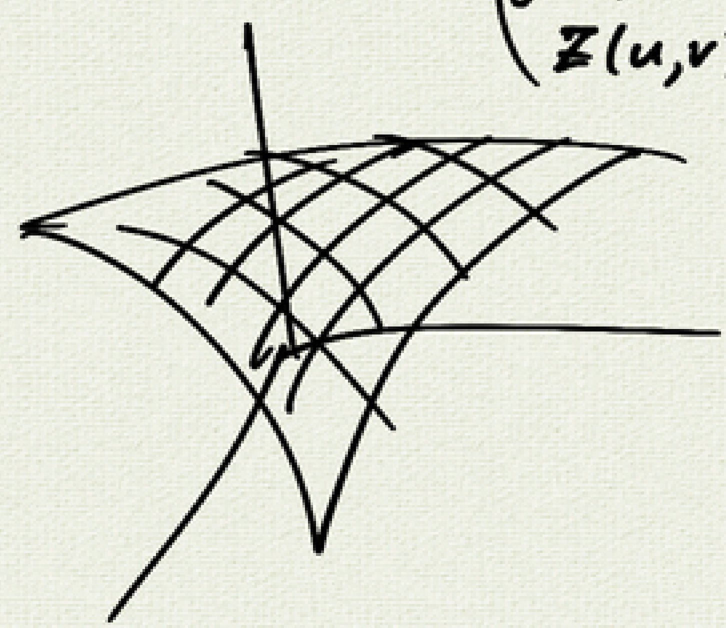
$\Rightarrow$ parametrize with 2 variables

$$\bar{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$$

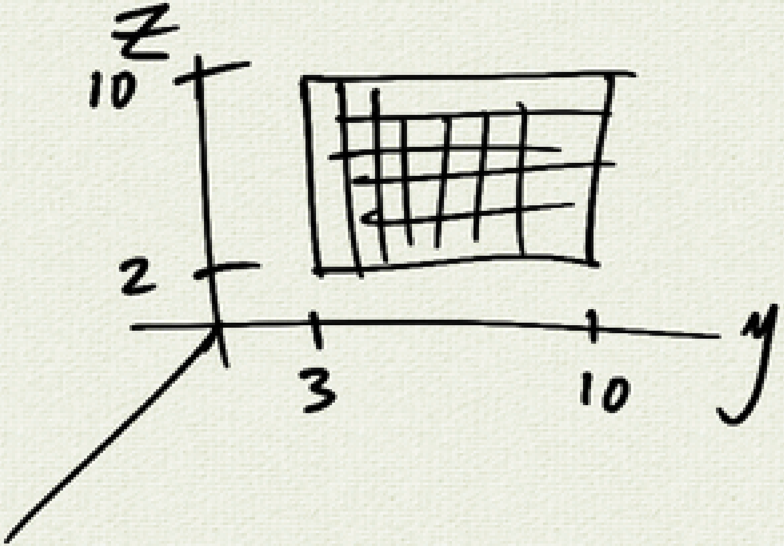
$$\bar{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



$\xrightarrow{\bar{r}}$



<u>curve</u> $\bar{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$



think: $u = y$

$v = z$

$x = 0$

$\Longrightarrow$

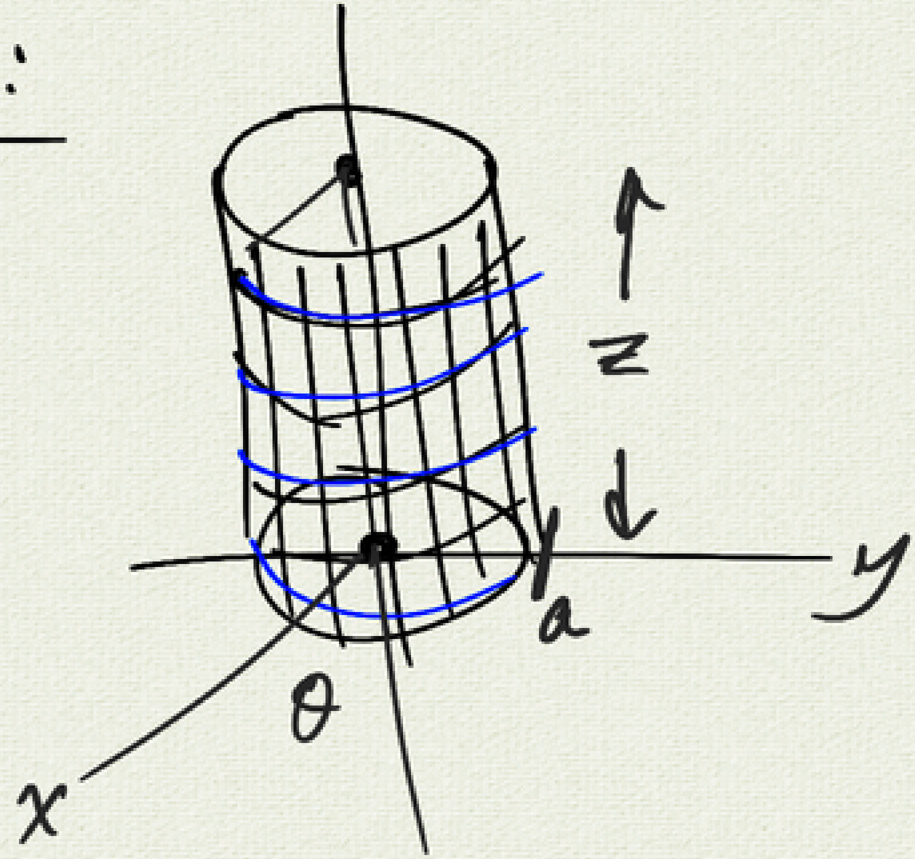
$$\bar{r}(u,v) = \begin{pmatrix} 0 \\ u \\ v \end{pmatrix}$$

$3 \leq u \leq 10$

$2 \leq v \leq 10$

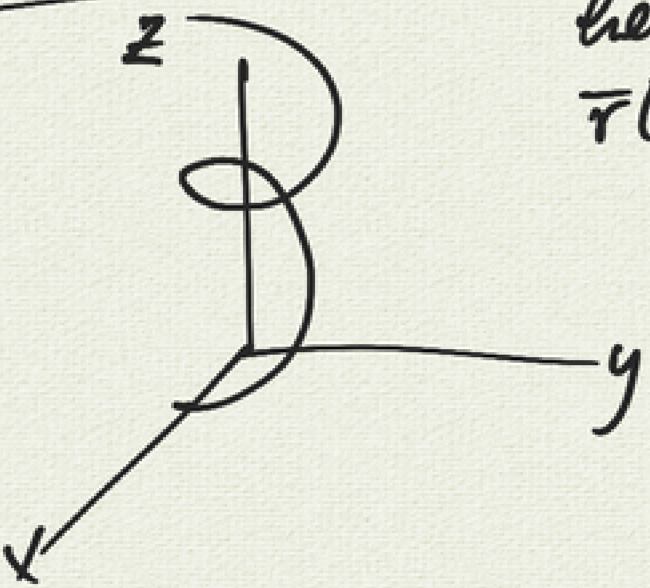
more natural $\Longrightarrow$ $$r(y,z) = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$$

cylinder :

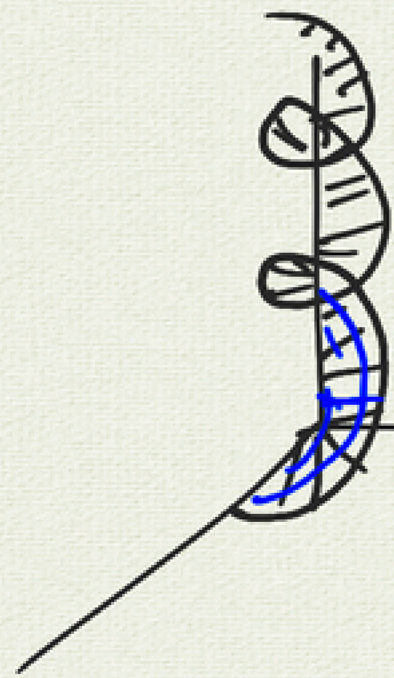


$$\bar{r}(\theta, z) = \begin{pmatrix} a\cos\theta \\ a\sin\theta \\ z \end{pmatrix}$$

$$\bar{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(\theta, z) \longmapsto (x, y, z)$$



helix
$$\bar{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

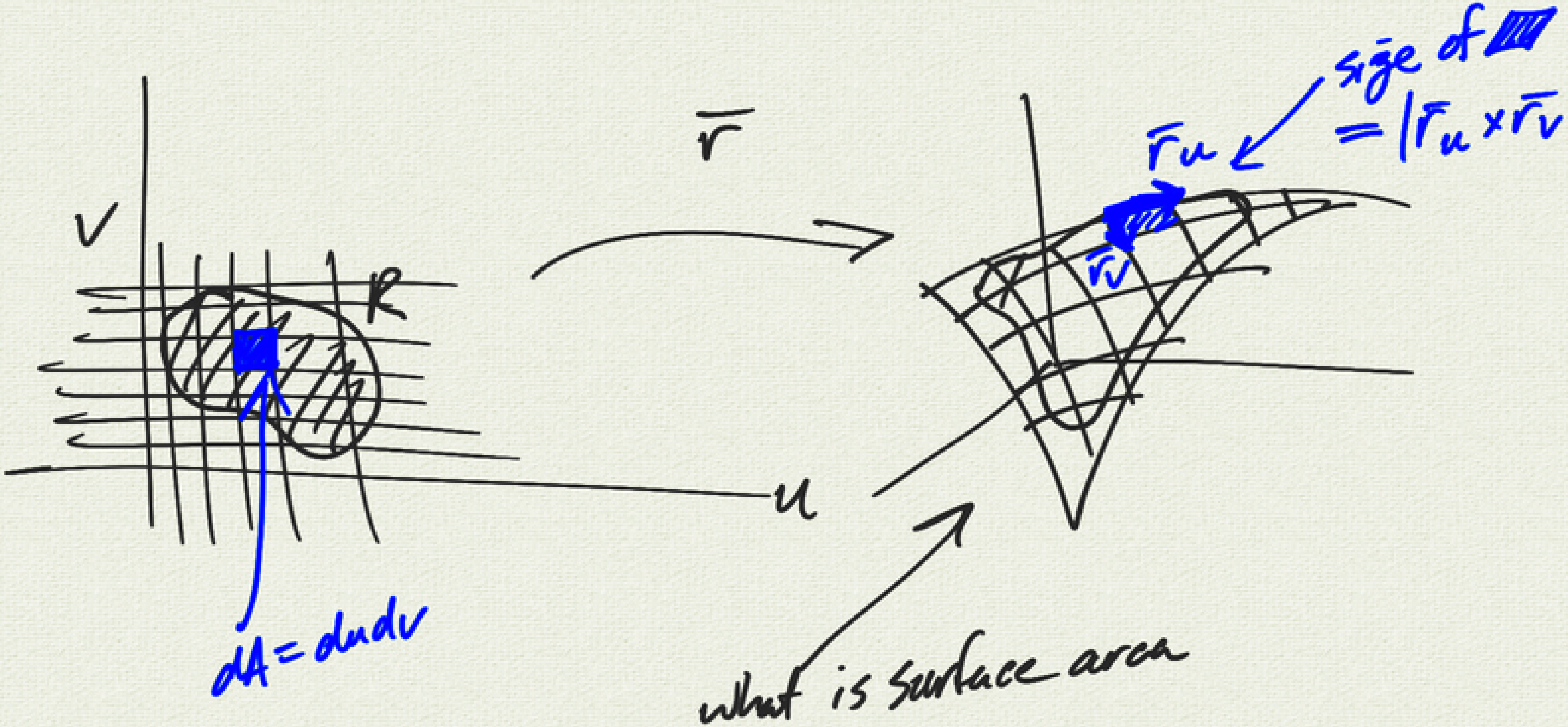


$$\bar{r}(u, v) = \begin{pmatrix} u\cos v \\ u\sin v \\ v \end{pmatrix}$$

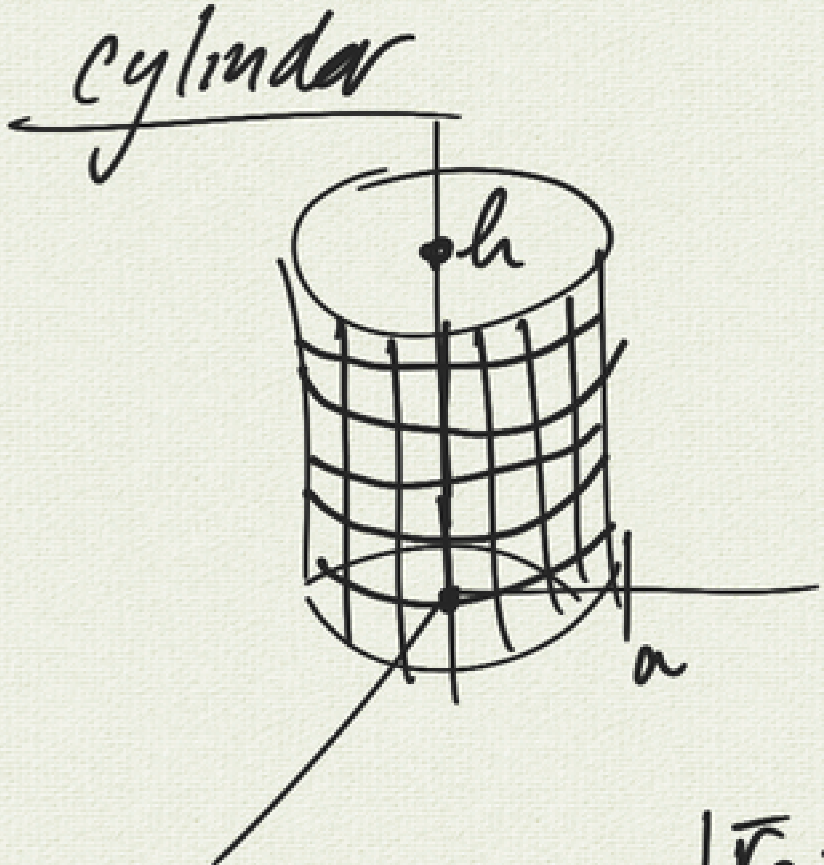
$0 \leq u \leq 1$

spiral pasta

helicoid



$$\boxed{SA = \iint_R |\bar{r}_u \times \bar{r}_v| \, du\, dv}$$



$$\bar{r}(\theta, z) = \begin{pmatrix} a\cos\theta \\ a\sin\theta \\ z \end{pmatrix}$$

$$SA = \iint_R |\bar{r}_\theta \times \bar{r}_z| \, d\theta \, dz$$

$$|\bar{r}_\theta \times \bar{r}_z| = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

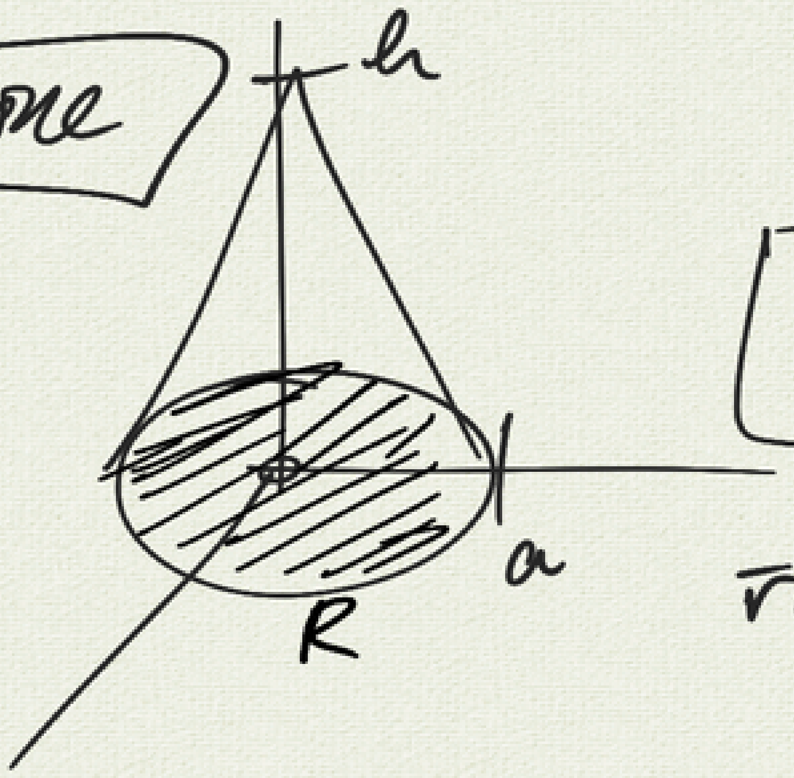
$$= |\langle a\cos\theta, +a\sin\theta, 0 \rangle|$$

$$= a$$

$$SA = \int_0^h \int_0^{2\pi} a \, d\theta \, dz$$

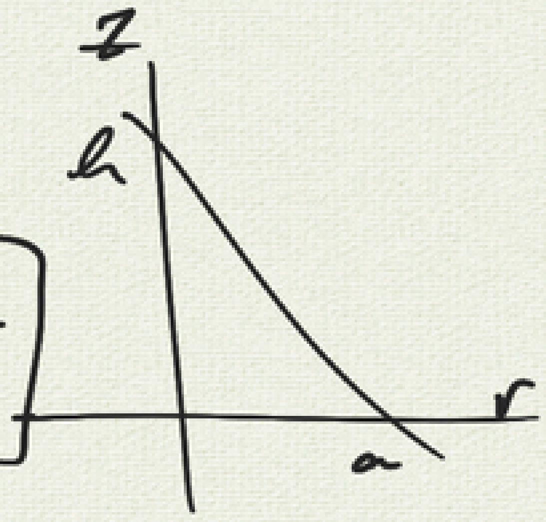
$$= 2\pi a h$$

Cone



$$z = h - \frac{h}{a} r$$

$$\boxed{z = h - \frac{h}{a}\sqrt{x^2+y^2}}$$



$$\bar{r}(x,y) = \begin{pmatrix} x \\ y \\ \underbrace{h - \frac{h}{a}\sqrt{x^2+y^2}}_{f(x,y)} \end{pmatrix}$$

$$f(x,y) = h - \frac{h}{a}\sqrt{x^2+y^2}$$

$$f_x = -\frac{h}{a}\,\frac{1}{2}\,\frac{2x}{\sqrt{x^2+y^2}}$$

$$= -\frac{h}{a}\,\frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = -\frac{h}{a}\,\frac{y}{\sqrt{x^2+y^2}}$$

$$|\bar{r}_x \times \bar{r}_y| = \left\| \begin{matrix} i & j & k \\ 1 & 0 & -\frac{h}{a}\frac{x}{r} \\ 0 & 1 & -\frac{h}{a}\frac{y}{r} \end{matrix} \right\|$$

$$= \left| \left\langle \frac{h}{a}\frac{x}{r}, \frac{h}{a}\frac{y}{r}, 1 \right\rangle \right|$$

$$= \sqrt{1 + \frac{h^2}{a^2}\left(\boxed{\frac{x^2+y^2}{r^2}}\right)}$$

$$= \sqrt{\frac{a^2+h^2}{a^2}} = \frac{l}{a} \Leftarrow \text{slant height}$$

$$SA = \iint_R |\bar{r}_x \times \bar{r}_y|\, dx\, dy$$

$$= \iint_R \frac{l}{a}\, dx\, dy$$

$$= \frac{l}{a} \underbrace{\iint_R dx\, dy}_{\pi a^2} \Leftarrow \text{area of circle radius } a$$

$$= \pi a l$$

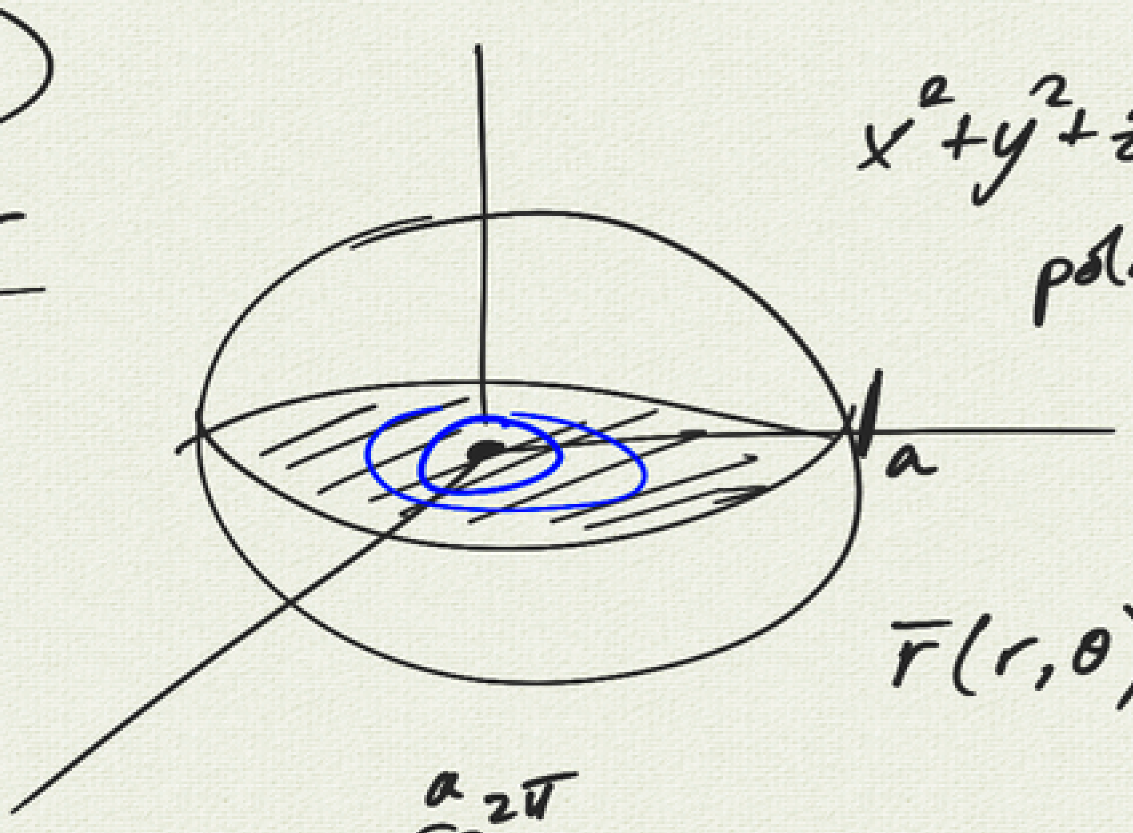
optional: do this in polar

$$\to |\bar{r}_r \times \bar{r}_\theta| = r\frac{l}{a}$$

$$\Rightarrow SA = \iint \frac{l}{a} r \cdot dr\, d\theta$$

Sphere

polar



$$x^2+y^2+z^2=a^2$$

polar: $x^2+y^2=r^2$

$$r^2+z^2=a^2$$

$$z=\sqrt{a^2-r^2}$$

$$\bar{r}(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{a^2-r^2} \end{pmatrix}$$

$$\Rightarrow SA=\int_0^a\int_0^{2\pi} |\bar{r}_r \times \bar{r}_\theta|\, d\theta\, dr$$

---

rect

$$\bar{r}(x,y)=\begin{pmatrix} x \\ y \\ \sqrt{a^2-x^2-y^2} \end{pmatrix}$$

$$\Rightarrow SA=\iint_{\{x^2+y^2\le a^2\}} |\bar{r}_x \times \bar{r}_y|\, dx\, dy$$

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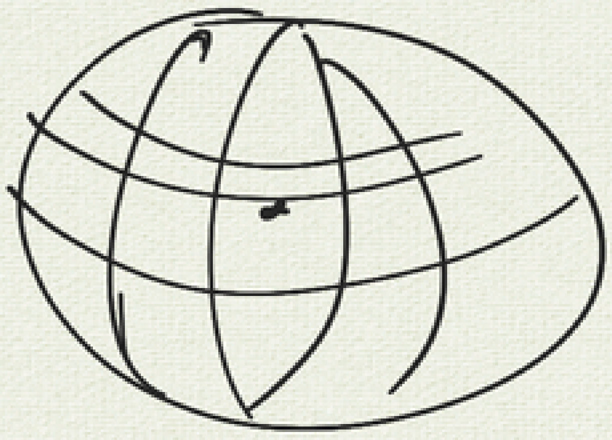
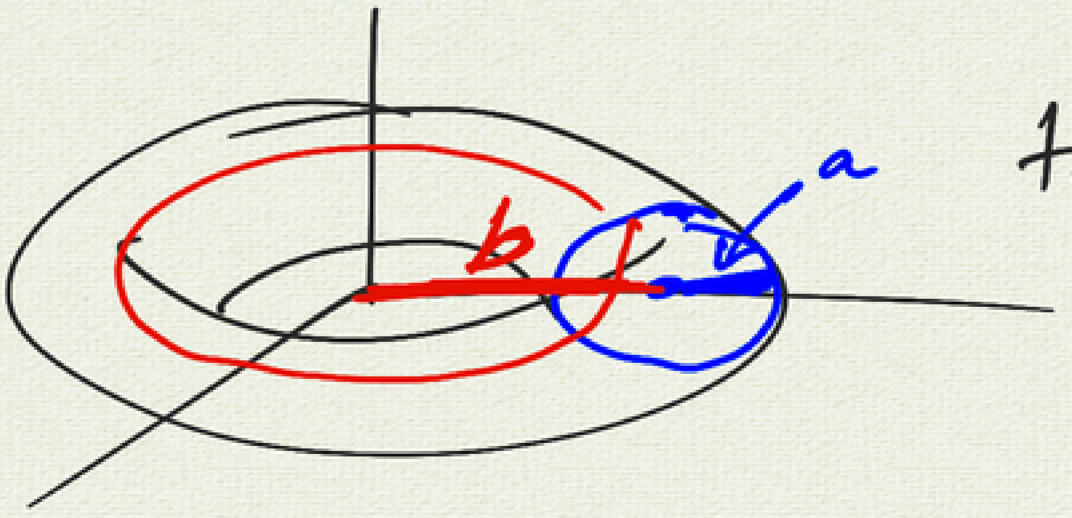
2 things to think about:



Spherical parametrization of surface of sphere



torus = donut

parametrize this!