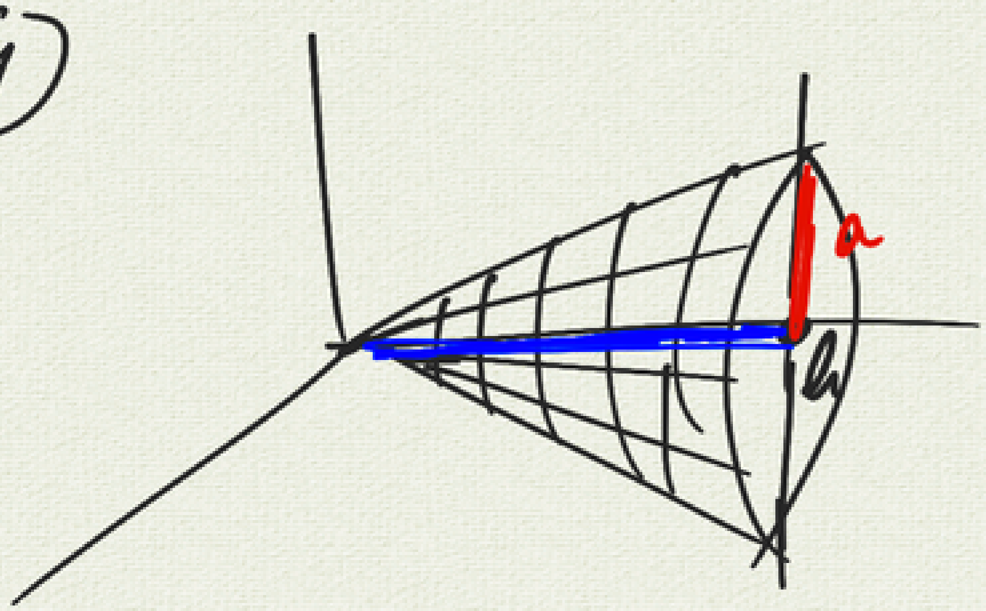
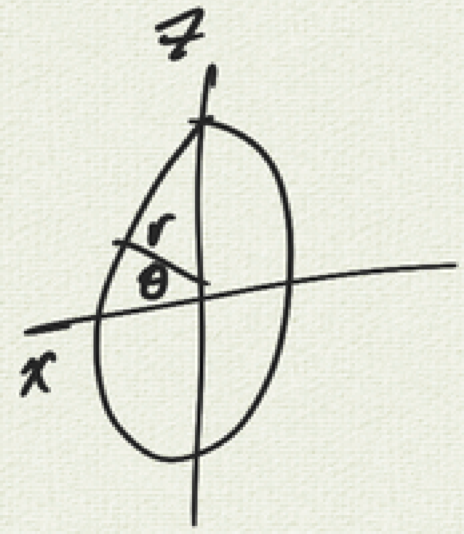
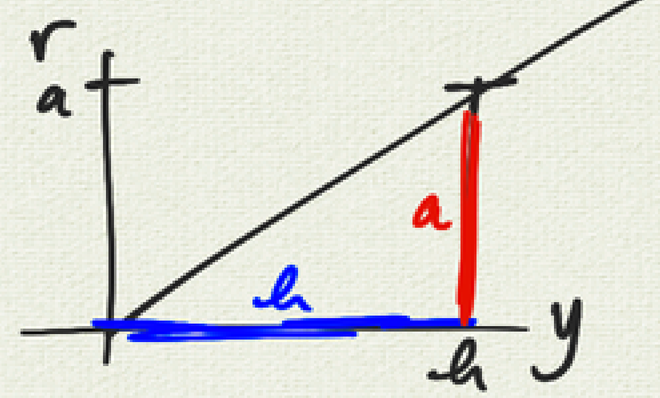
15.4 Guichard

(1)





$$\boxed{\begin{aligned} x &= r\cos\theta \\ z &= r\sin\theta \end{aligned}}$$



$$r = \frac{a}{h} y \implies \boxed{y = \frac{h}{a} r}$$

$$\vec{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ \frac{h}{a} r \\ r\sin\theta \end{pmatrix}$$

$$\implies |\vec{r}_r \times \vec{r}_\theta| = r \underbrace{\sqrt{\left(\frac{h}{a}\right)^2 + 1}}_{l}$$

l slant height

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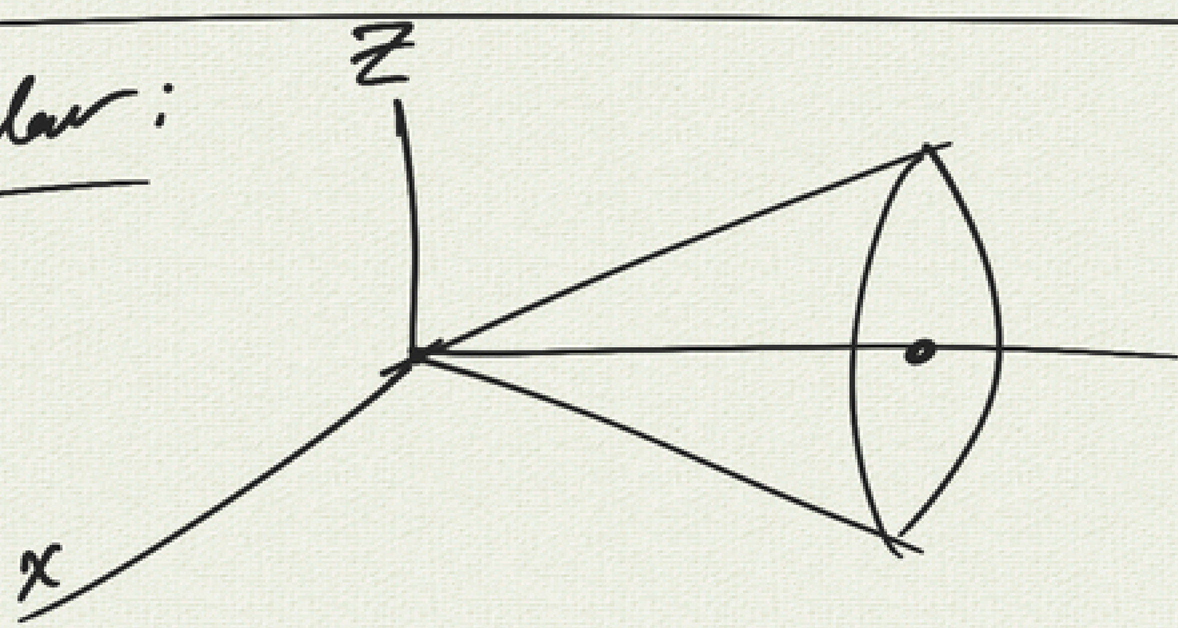
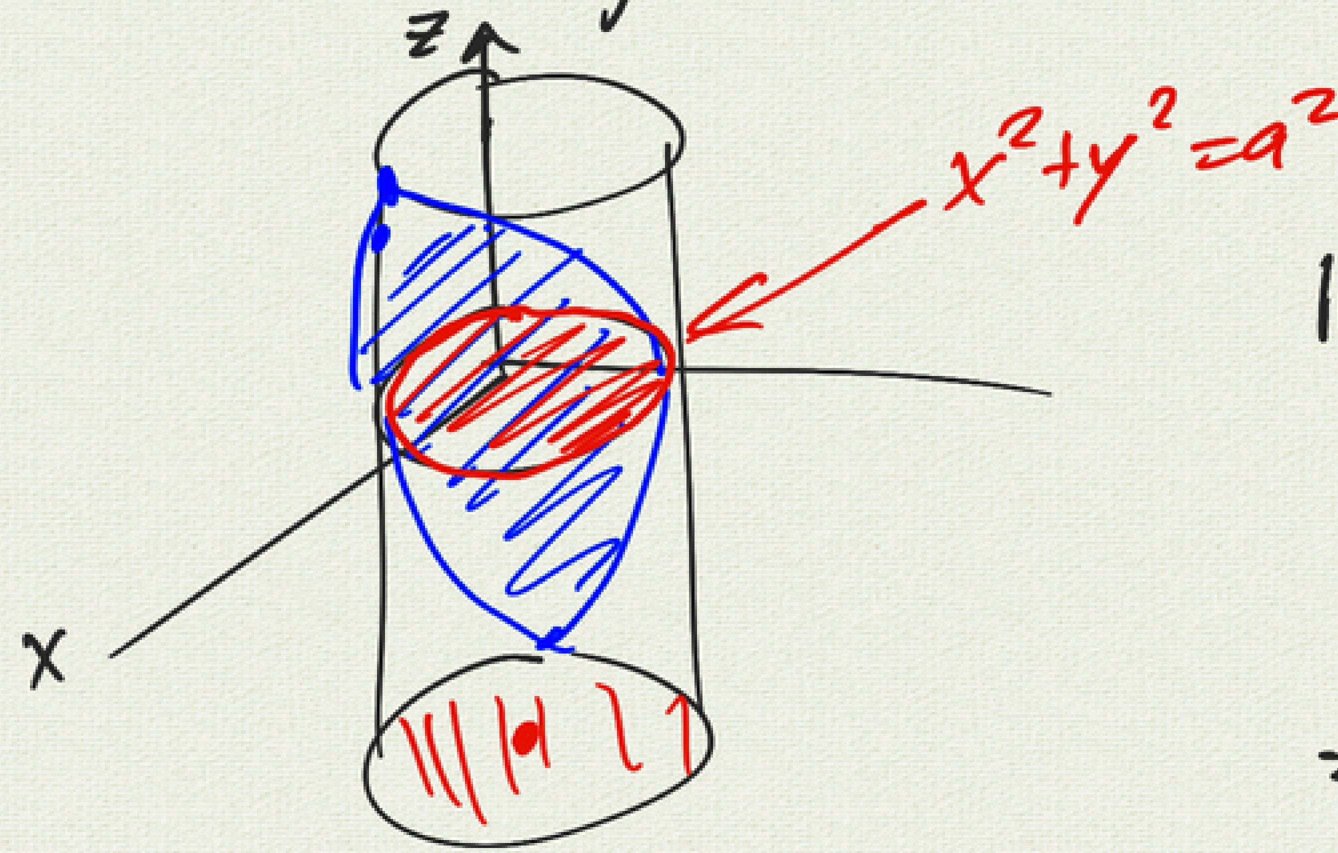
rectangular:



$$\begin{pmatrix} x \\ \frac{h}{a}\sqrt{x^2+z^2} \\ z \end{pmatrix}$$

(2) $z = mx$

inside $x^2+y^2=a^2$



$$\bar{r}(x,y) = \begin{pmatrix} x \\ y \\ mx \end{pmatrix}$$

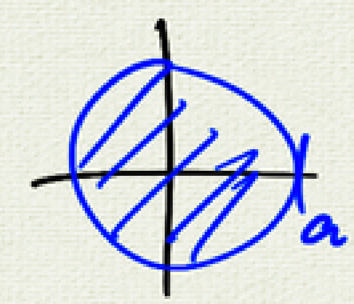
$$|\bar{r}_x \times \bar{r}_y| = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & m \\ 0 & 1 & 0 \end{matrix} \right|$$

$$= |\langle -m, 0, 1 \rangle|$$

$$= \sqrt{1+m^2}$$

$$SA = \iint_{x^2+y^2 \le a^2} |\bar{r}_x \times \bar{r}_y| \, dx\,dy$$

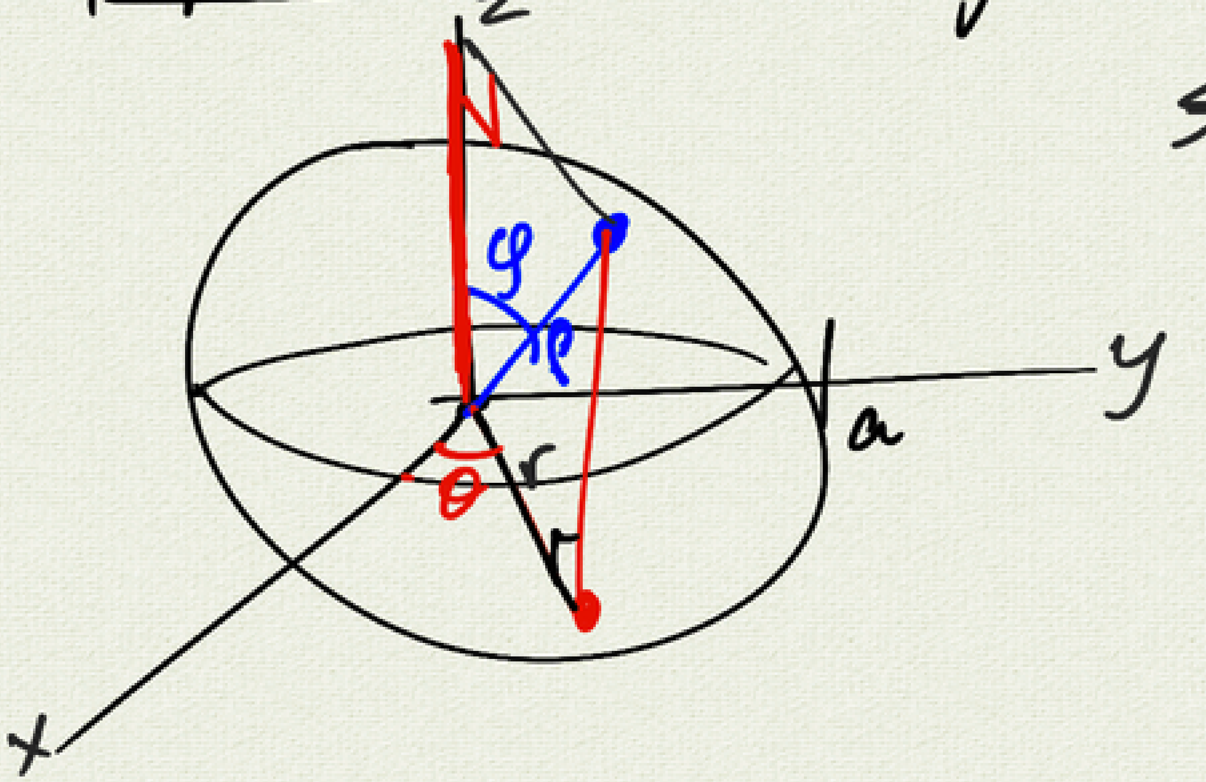
$$= \sqrt{1+m^2} \boxed{\iint_{x^2+y^2 \le a^2} dx\,dy}$$



$$= \pi a^2 \sqrt{1+m^2}$$

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

Sphere



$x^2+y^2+z^2=a^2$

spherical: (ρ, θ, φ)

$z = \rho\cos\varphi$

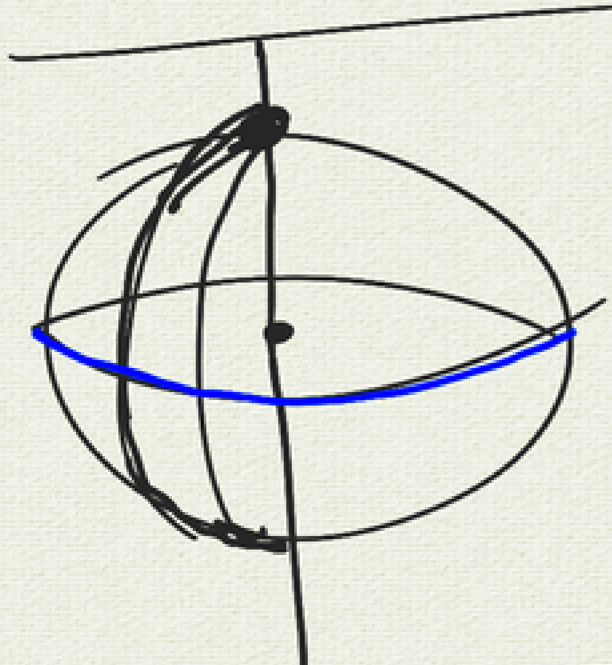
$r = \rho\sin\varphi$

$x = r\cos\theta$
$= (\rho\sin\varphi)\cos\theta$

$y = r\sin\theta$
$= (\rho\sin\varphi)\sin\theta$

$\Longrightarrow$ sphere
$\rho = a$

$$\bar{r}(\theta,\varphi) = \begin{pmatrix} a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \\ a\cos\varphi \end{pmatrix}$$



longitude $[0, 2\pi]$

latitude $[-\frac{\pi}{2}, \frac{\pi}{2}]$

spherical

$\theta : [0, 2\pi]$

$\varphi : [0, \pi]$

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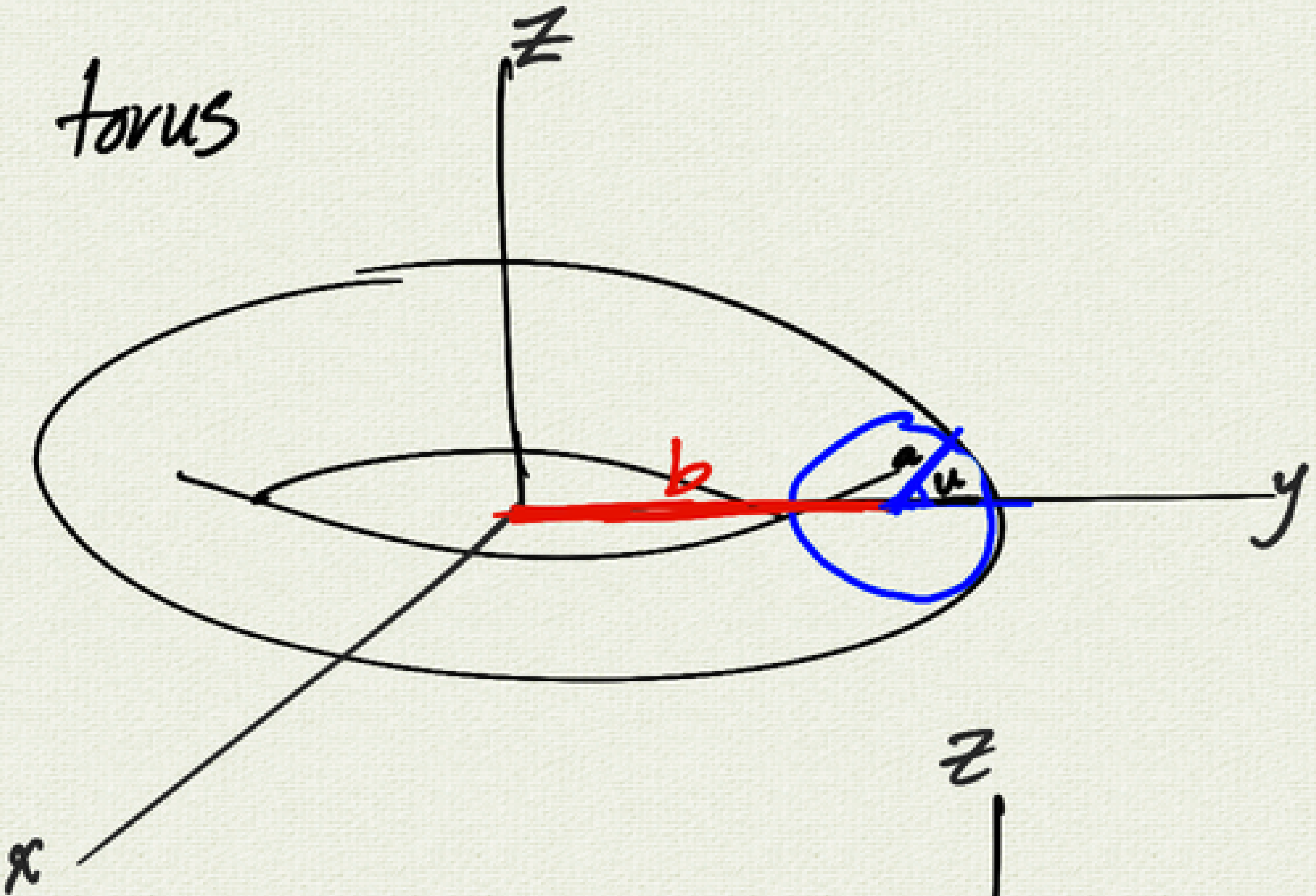
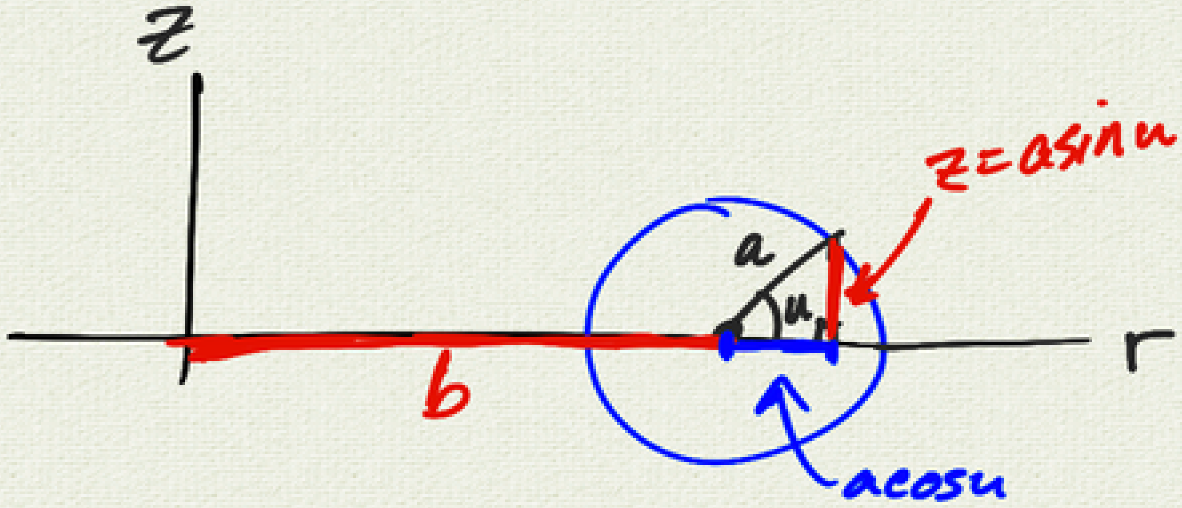
exercise: $|\bar{r}_\theta \times \bar{r}_\varphi| = a^2\sin\varphi$

torus



$0 \le u \le 2\pi$

$z = a\sin u$



$r = b + a\cos u$

think:
$x = r\cos v$
$y = r\sin v$
$\uparrow$
θ

$$\Rightarrow \quad \bar{r}(u,v) = \begin{pmatrix} (b+a\cos u)\cos v \\ (b+a\cos u)\sin v \\ a\sin u \end{pmatrix}$$

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exercise: $|\bar{r}_u \times r_v| = a(b + a\cos u)$

$$SA = \int_0^{2\pi}\!\!\int_0^{2\pi} |\bar{r}_u \times \bar{r}_v| \, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} ab\, du\, dv + \underbrace{\int_0^{2\pi}\int_0^{2\pi} a^2\cos u\, du\, dv}_{0}$$

$$= (2\pi a)(2\pi b)$$

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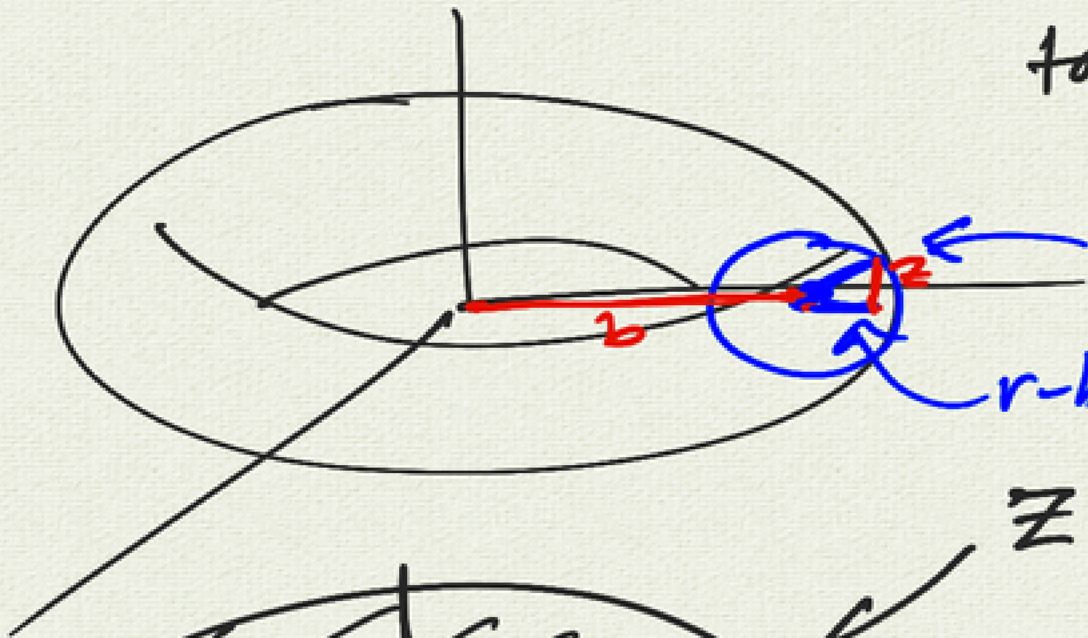
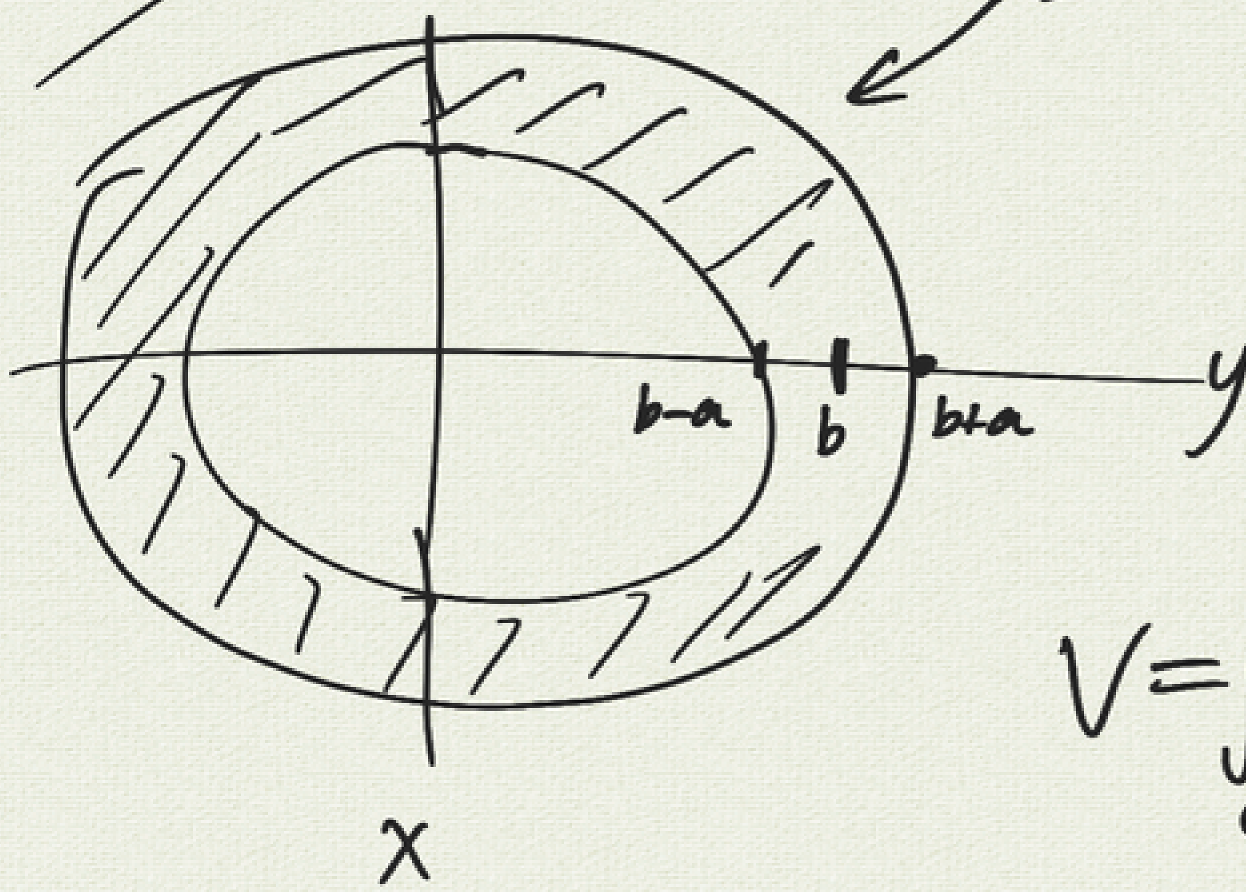
torus volume



$$z^2+(r-b)^2=a^2$$

r-b

$$z=\sqrt{a^2-(r-b)^2}$$



$$V=\int_0^{2\pi}\int_{b-a}^{b+a} 2\sqrt{a^2-(r-b)^2}\; r\,dr\,d\theta$$

tricky but doable

sub $\boxed{u=r-b}$

$$V=(\pi a^2)(2\pi b)$$

area of small circle

circumference of big circle

think about: why is the same as cylindrial shells?