OSC3 5.5

269

$f(x,y,z)=1$

$x^2+y^2+z^2 \le 90$

$\Rightarrow$

253 $\quad f(x,y,z) = \dfrac{1}{x+3}$

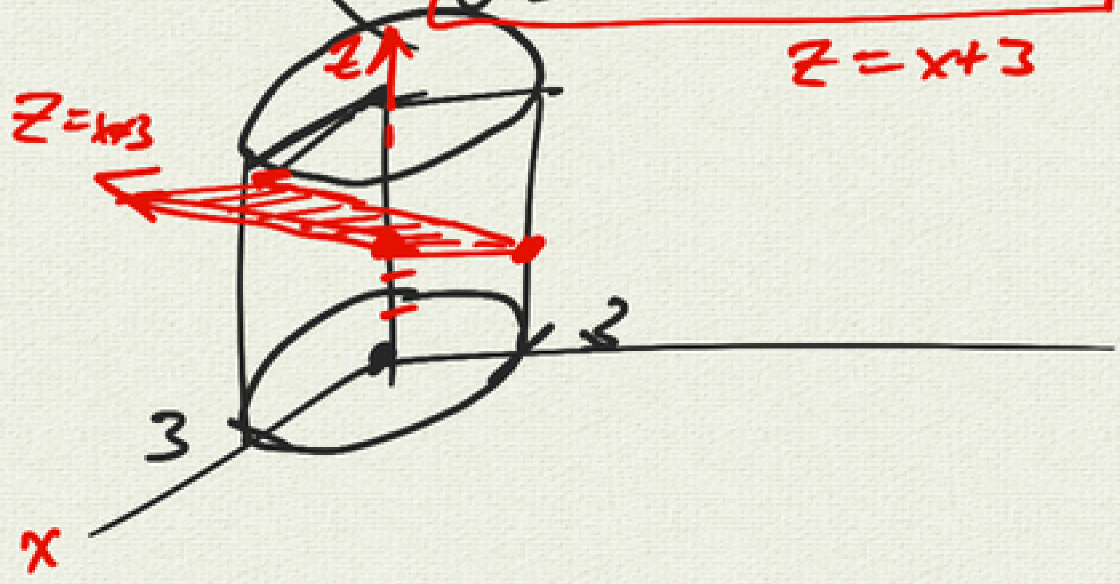
$$\iiint_E f(x,y,z)\,dV$$

$dx\,dy\,dz$

$r\,dr\,d\theta\,dz$

$$E: \begin{cases} x^2+y^2 \le 9 \\ x \ge 0 \\ y \ge 0 \\ \boxed{0 \le z \le x+3} \end{cases}$$

$z = x+3$



$$\iiint f\,dV$$

$$= \int_0^{\pi/2}\int_0^3\int_0^{r\cos\theta+3} \left[\frac{1}{r\cos\theta+3}\right] r\,dz\,dr\,d\theta$$

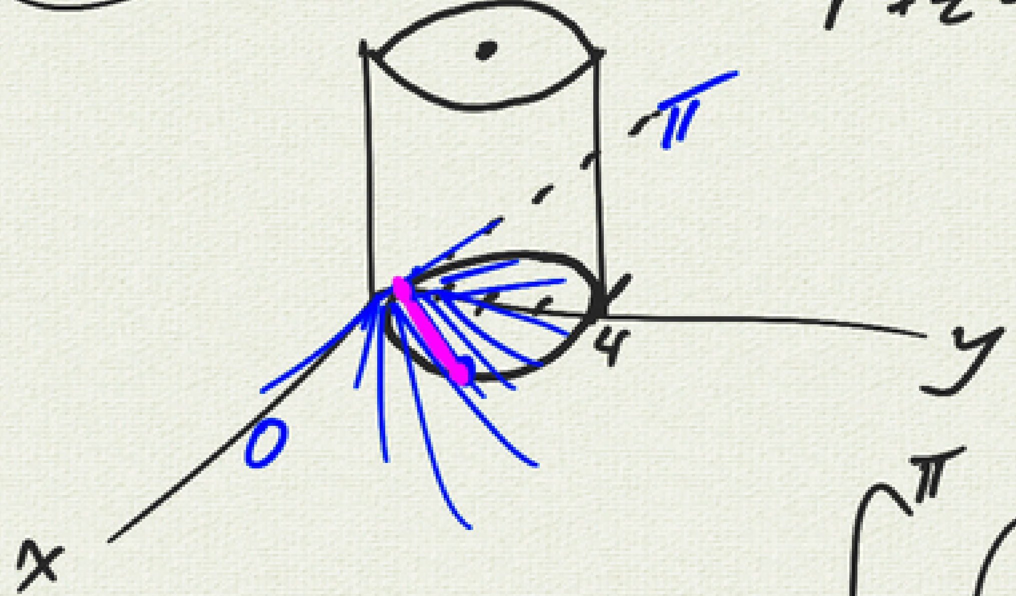
$$= \int_0^{\pi/2}\int_0^3 r\,dr\,d\theta$$

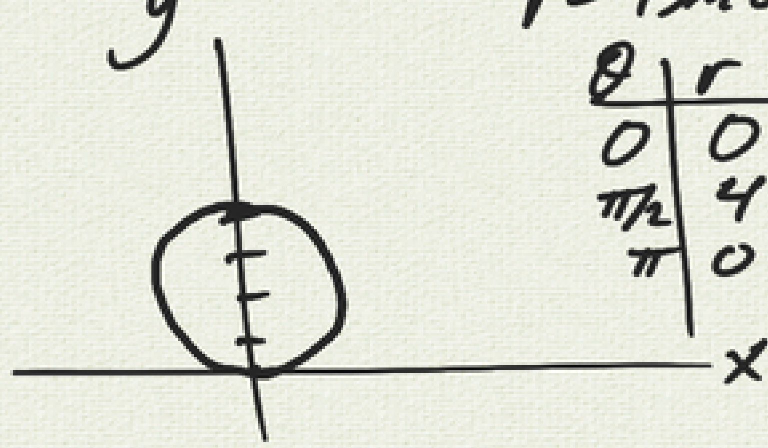
$$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^3 d\theta$$

$$= \frac{9}{2}\frac{\pi}{2} = \frac{9\pi}{4}$$

249 $r = 4\sin\theta$

under sphere $r^2 + z^2 = 16$





$r = 4\sin\theta$

θ	r
0	0
$\pi/2$	4
π	0

$$\int_0^{\pi}\int_0^{4\sin\theta}\int_0^{\sqrt{16-r^2}} f(x,y,z)\, r \; dz\, dr\, d\theta$$

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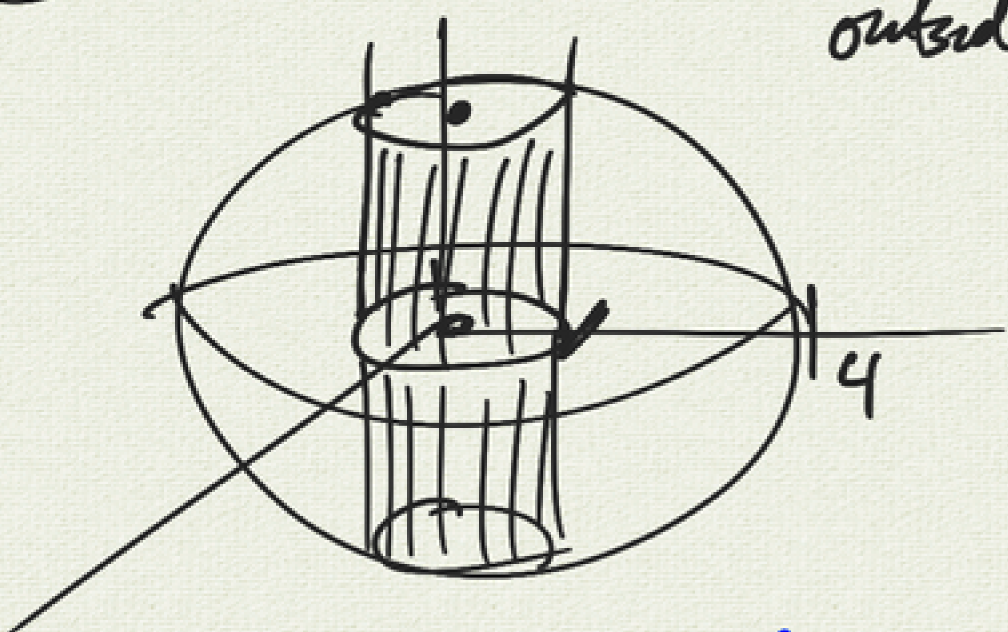
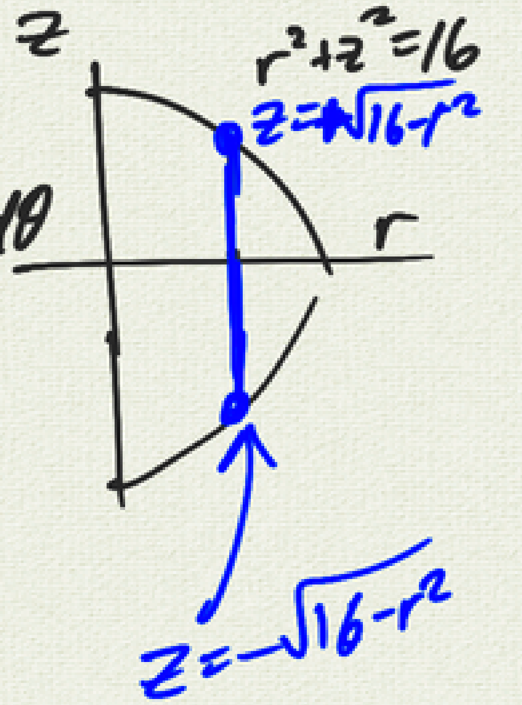
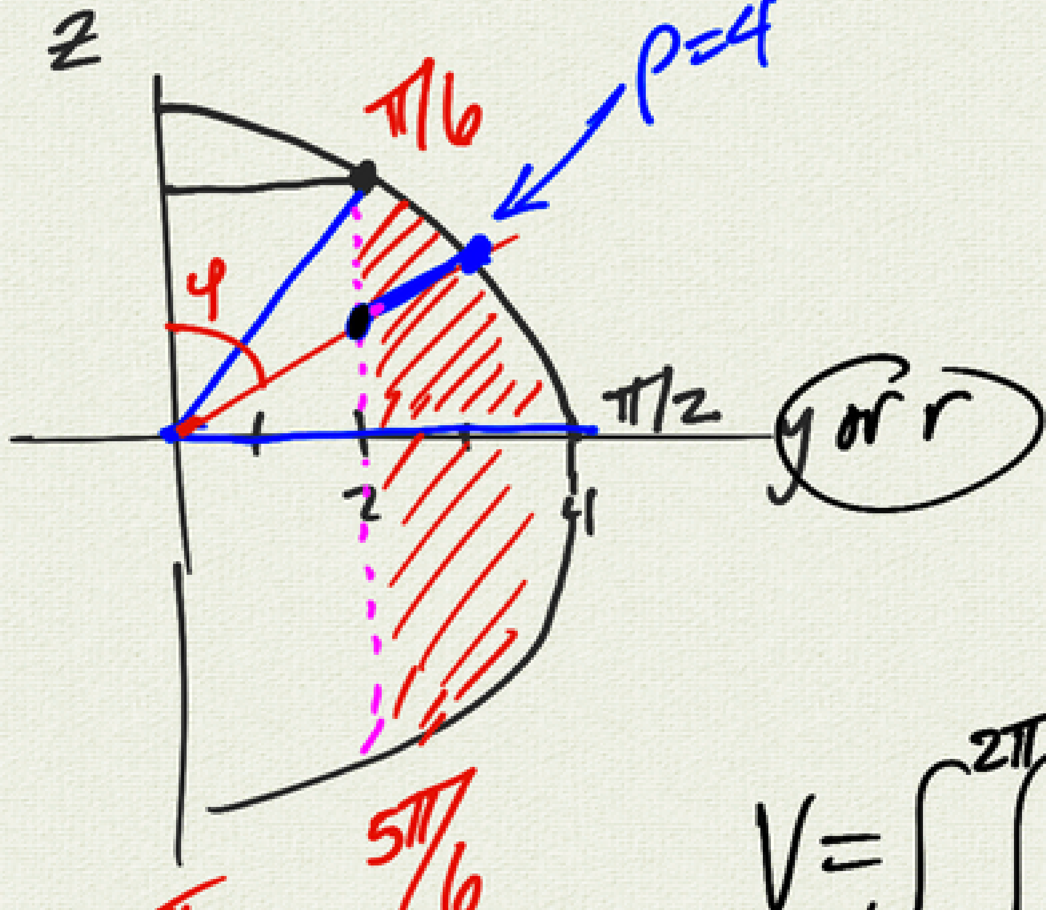
291 inside sphere $x^2+y^2+z^2=16$ ⇐

outside cylinder $x^2+y^2=4$



$$V = \int_0^{2\pi}\int_2^{4}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta$$



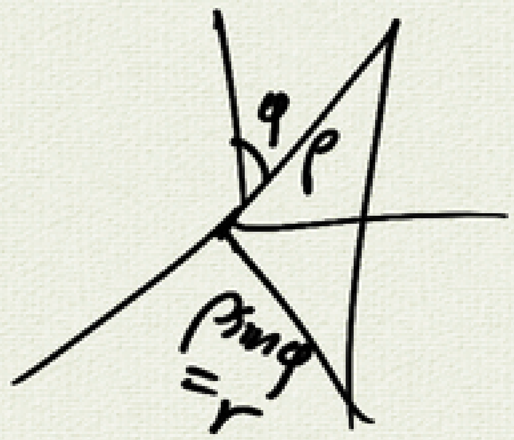


$r = \rho\sin\varphi$

$r = 2$

$\Rightarrow \rho\sin\varphi = 2$

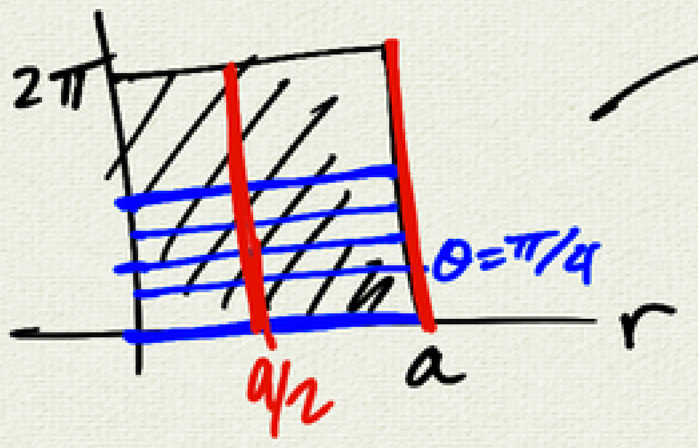
$\rho = \dfrac{2}{\sin\varphi}$



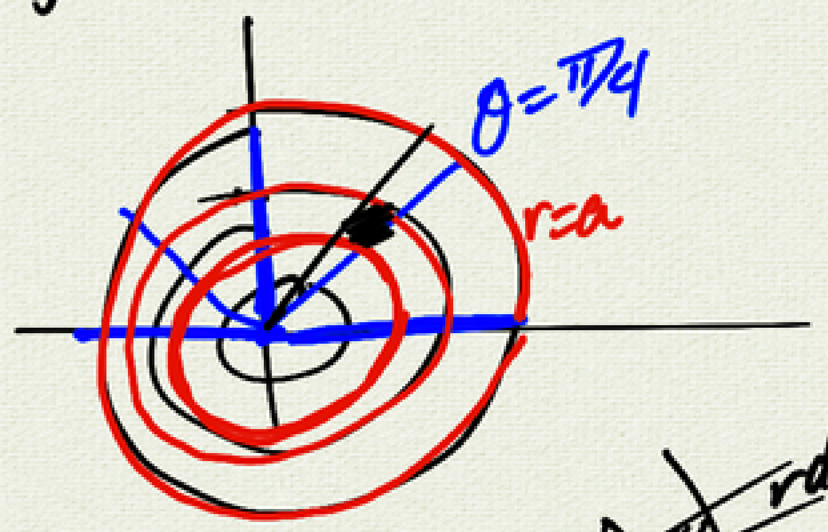
$$V = \int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_0^{\frac{2}{\sin\varphi}} \left(\rho^2\sin\varphi\right) d\rho\, d\varphi\, d\theta$$

5.9 Change of Variables

<u>polar</u>



$$\vec{r}(r,\theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

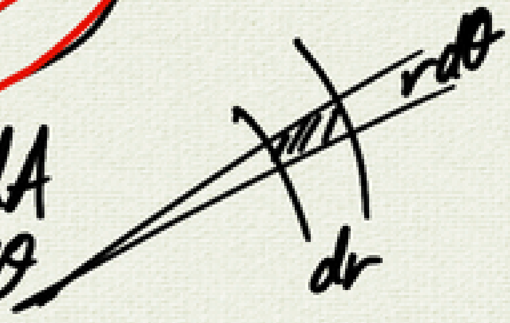


$x(r,\theta) = r\cos\theta$

$$\Rightarrow \Delta x \approx \frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial \theta}\Delta\theta$$

$$\Delta y \approx \frac{\partial y}{\partial r}\Delta r + \frac{\partial y}{\partial \theta}\Delta\theta$$

$dA = r\,dr\,d\theta$



$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \approx \underbrace{\boxed{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}}}_{D} \begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix}$$

D linear transformation / derivative

$$dA = dx\,dy = \boxed{|\det D|}\,dr\,d\theta$$

notation $\dfrac{\partial(x,y)}{\partial(r,\theta)}$

D Jacobian (derivative)

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

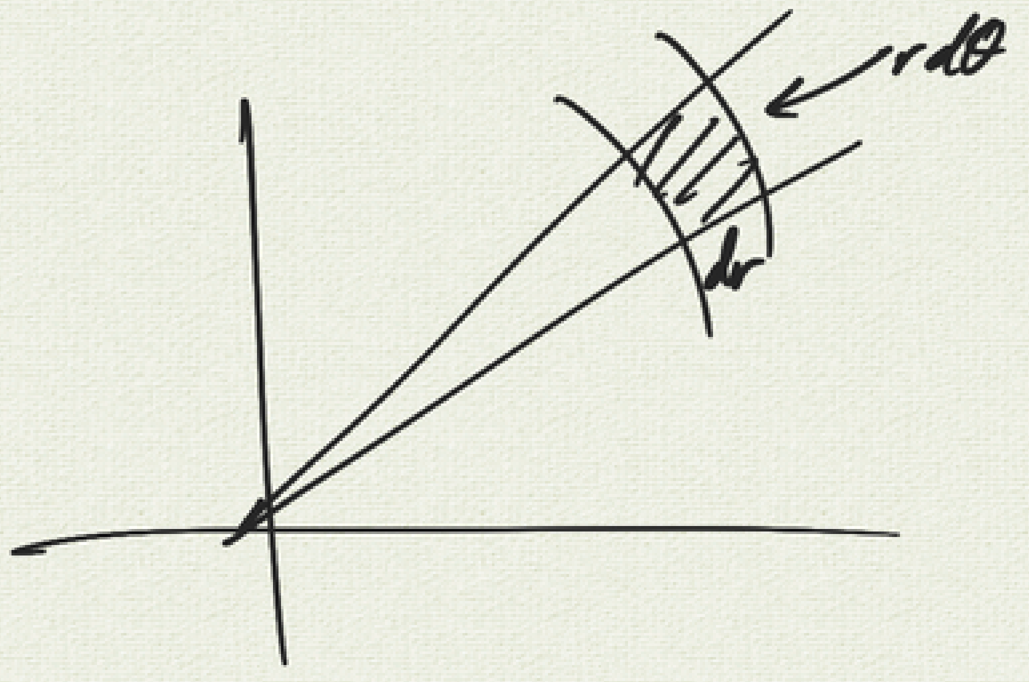
$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

determinant + absolute value

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\Rightarrow \quad dA = dx\,dy = \underbrace{r}_{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}\,dr\,d\theta$$



$\underline{\text{cylindrical}}$ $(r, \theta, z) \to (x, y, z)$

$$\iiint \underbrace{dx\,dy\,dz}_{dV} = \iiint \underbrace{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|} dr\,d\theta\,dz$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$r(r, \theta, z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

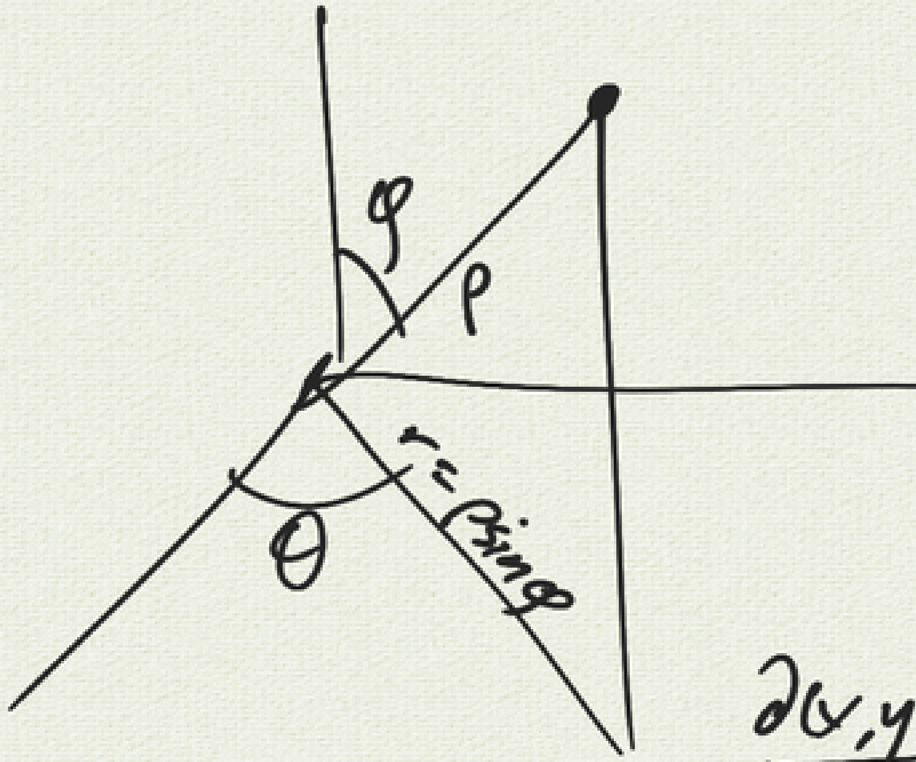
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

$$dV = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz$$

$$= r\,dr\,d\theta\,dz$$

Spherical (ρ, θ, φ)



$$z = \rho\cos\varphi$$
$$r = \rho\sin\varphi$$

$$\vec{r}(\rho,\theta,\varphi) = \begin{pmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{pmatrix}$$

$$\iiint_V f \underset{dx\,dy\,dz}{dV} = \iiint \underbrace{\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right|}_{\rho^2\sin\varphi} d\rho\, d\theta\, d\varphi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\right) - \rho\sin\varphi\left(\rho\sin^2\varphi\right)$$

$$= -\rho^2\sin\varphi\,(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1})$$

$$\Longrightarrow \quad dV = \boxed{\rho^2\sin\varphi}\, d\rho\, d\theta\, d\varphi$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right|$$