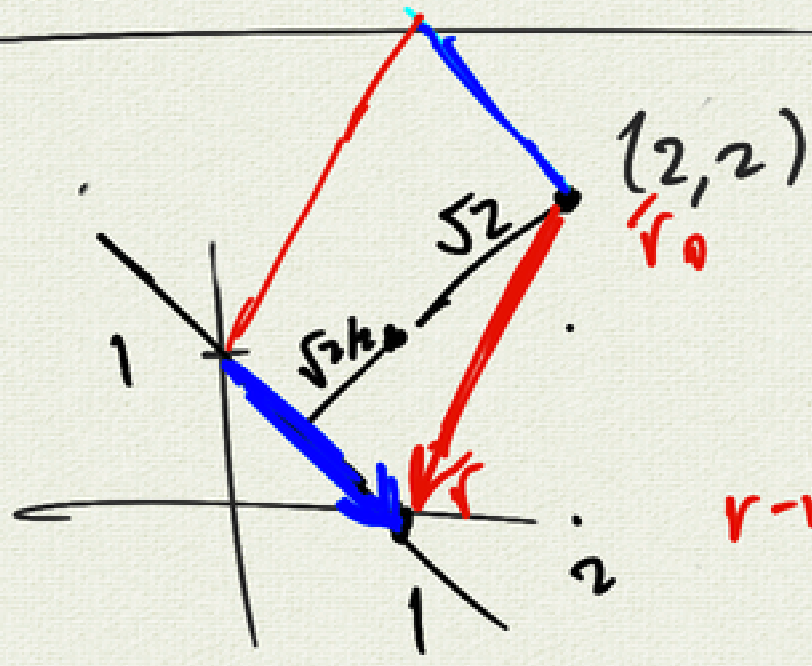
6.2 Geometric Product

(3)



$$r-r_0=\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$v=\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow d=\frac{|v\wedge(r-r_0)|}{|v|}$$

$$|v\wedge(r-r_0)|=\left|\begin{vmatrix}1 & -1\\-1 & -2\end{vmatrix}\right|e_1\wedge e_2$$

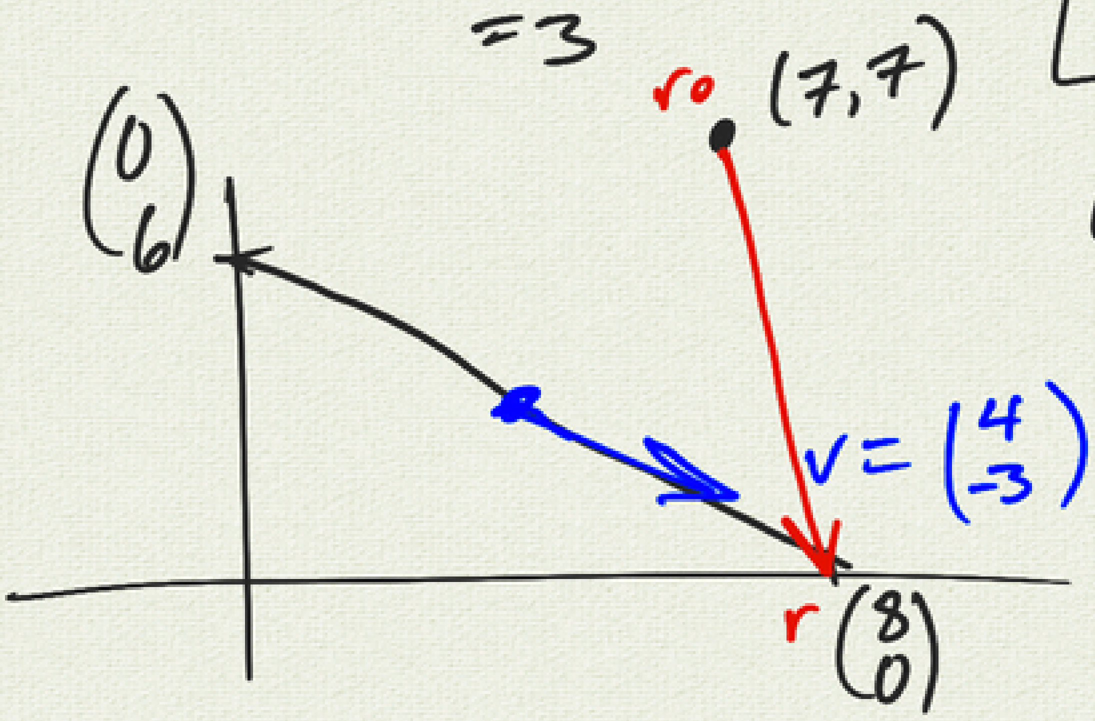
$$=|-3|$$

$$=3$$

$$=\frac{3}{\sqrt{2}}$$

$$\boxed{d=\frac{3\sqrt{2}}{2}}$$

(4)



$$6x+8y=48$$

$$r-r_0=\begin{pmatrix}1\\-7\end{pmatrix}$$

$$\Rightarrow \|v\wedge(r-r_0)\|$$

$$=\left\|\begin{matrix}4 & 1\\-3 & -7\end{matrix}\right\|$$

$$=25$$

$$d=\frac{25}{|v|}=\frac{25}{5}=5$$

6.2 Geometric Product

wedge product: $e_1 \wedge e_1 = 0$ $\qquad$ $e_1 \wedge e_2$ unit bivector ($1\ \text{unit}^2$)

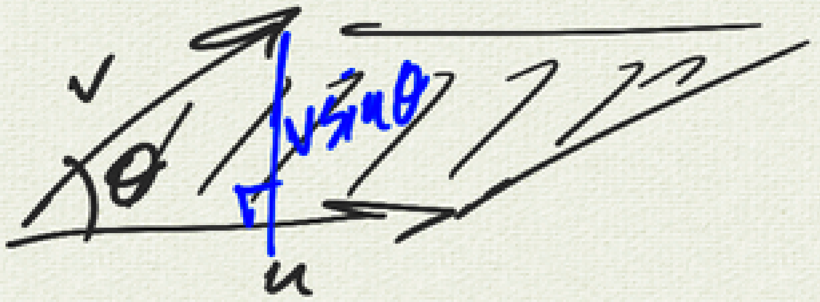
$u \wedge u = 0$

$u \wedge v = -v \wedge u$ antisymmetric

FOIL: $(ae_1 + be_2) \wedge (ce_1 + de_2)$ $\quad\Big|\quad$ $\begin{pmatrix} a \\ b \end{pmatrix} \wedge \begin{pmatrix} c \\ d \end{pmatrix}$

$$= ac\, \underline{e_1 \wedge e_1} + ad(e_1 \wedge e_2)$$
$$bc(e_2 \wedge e_1) + bd\, \underline{e_2 \wedge e_2}$$

$$= (ad - bc)\, e_1 \wedge e_2$$



$|u \wedge v| = |u||v|\sin\theta$

dot product

$$u = \begin{pmatrix} a \\ b \end{pmatrix} \quad v = \begin{pmatrix} c \\ d \end{pmatrix}$$

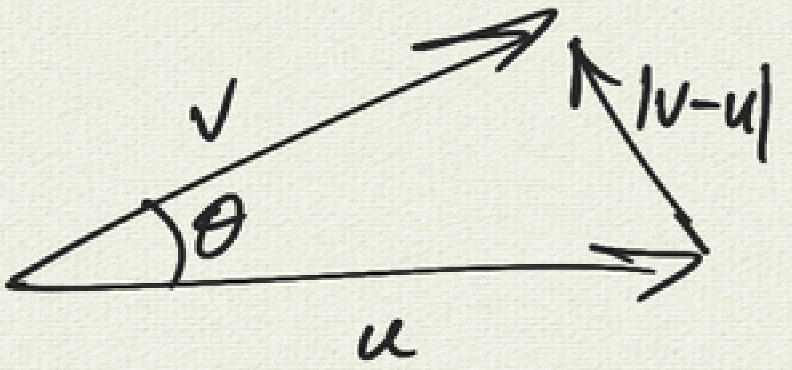
$$u \cdot v = ac + bd \quad \underline{\underline{\text{scalar}}}$$

$$u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$u \cdot v = x_1 x_2 + y_1 y_2$$

$$e_1 \cdot e_1 = 1 = e_2 \cdot e_2$$

$$u \cdot u = a^2 + b^2 = |u|^2$$



Law of Cosines

$$|v-u|^2 = \boxed{|u|^2 + |v|^2 - 2|u||v|\cos\theta}$$

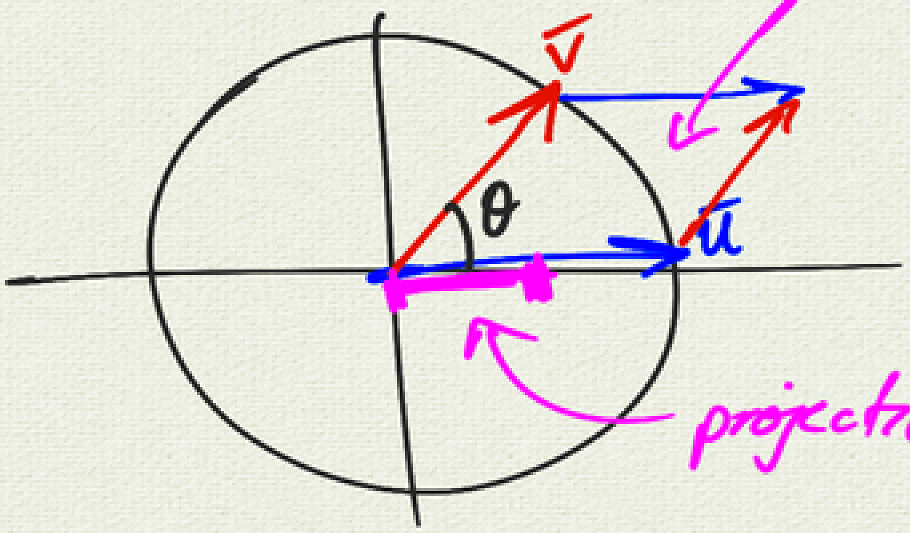
FOIL:

$$\begin{aligned} |v-u|^2 &= (v-u)\cdot(v-u) \\ &= v\cdot v - v\cdot u - u\cdot v + u\cdot u \\ &= \boxed{|u|^2 + |v|^2 - 2u\cdot v} \end{aligned}$$

$$\Longrightarrow u \cdot v = |u||v|\cos\theta$$

$$|u \wedge v| = |u||v|\sin\theta$$

area $= \sin\theta$
$= |u \wedge v|$



projection $= \cos\theta$
$= u \cdot v$

define:

$$\underbrace{uv}_{\text{geometric product}} = \underbrace{u \cdot v}_{\substack{\text{dot product}\\ \text{scalar}}} + \underbrace{u \wedge v}_{\substack{\text{wedge product}\\ \text{bivector (area)}}}$$

vectors

$$uu = u \cdot u + \underbrace{u \wedge u}_{0}$$

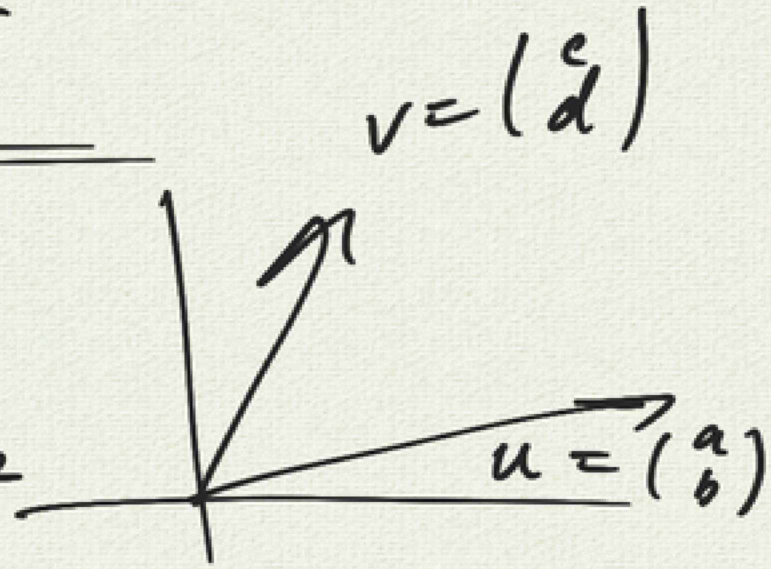
$$u^2 = |u|^2$$

$$e_1^2 = 1 = e_2^2$$

$$e_1 e_2 = \underbrace{e_1 \cdot e_2}_{0} + e_1 \wedge e_2$$

$$\boxed{e_1 e_2 = e_1 \wedge e_2}$$ unit bivector

$u = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad v = \begin{pmatrix} c \\ d \end{pmatrix}$



$$uv = u \cdot v + u \wedge v$$
$$= \underbrace{(ac+bd)}_{\text{scalar}} + \underbrace{(ad-bc)e_1 \wedge e_2}_{\text{bivector}}$$

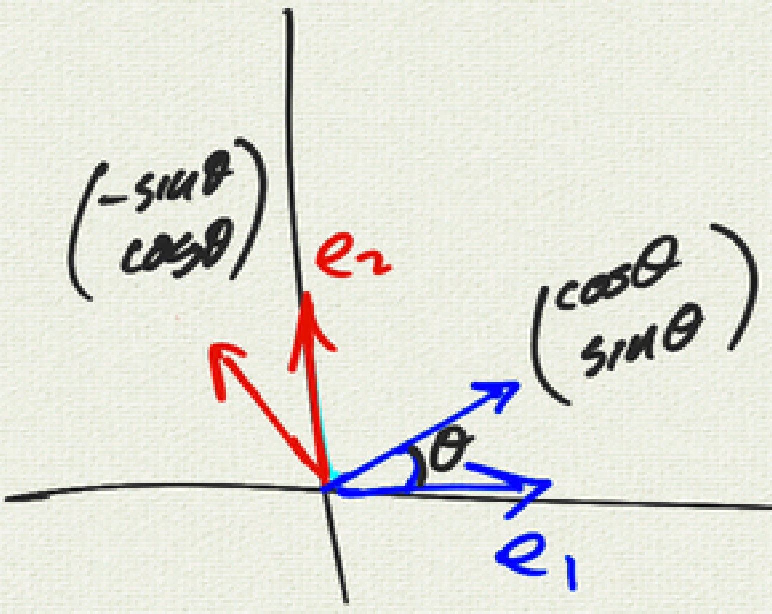
$e_1^2 = 1 = e_2^2$

$$uv = (ae_1 + be_2)(ce_1 + de_2)$$
$$= ac\underbrace{e_1 e_1}_{1} + ad\,e_1 e_2 + bc\,e_2 e_1 + bd\underbrace{e_2 e_2}_{1}$$
$$= (ac+bd) + (ad-bc)e_1 e_2$$

$$u^2 = |u|^2 \implies \text{define } u^{-1} = \frac{u}{|u|^2}$$

inverse of a vector

rotations in $\mathbb{R}^2$



$$\begin{array}{cc} & \begin{array}{cc} e_1 & \quad e_2 \end{array} \\ R_\theta = & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{array}$$

vector $w = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$

$\Rightarrow$ rotate by θ

$$R_\theta w = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix}$$

$$= \begin{pmatrix} w_x\cos\theta - w_y\sin\theta \\ w_x\sin\theta + w_y\cos\theta \end{pmatrix}$$

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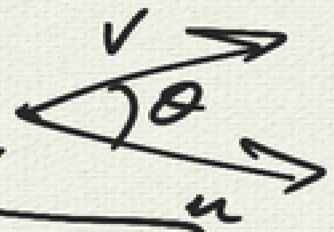
u, v unit vectors



$u\cdot v = |u||v|\cos\theta$

$|u\wedge v| = |u||v|\sin\theta$

$\Rightarrow uv = u\cdot v + u\wedge v$

$= \boxed{\cos\theta + \sin\theta(e_1e_2)}$ weird

scalar

bivector

$w = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$

$$w(uv) = (w_x e_1 + w_y e_2)(\cos\theta + \sin\theta\, e_1e_2)$$

$$= w_x\cos\theta\, e_1 + w_x\sin\theta\, \underbrace{\underbrace{e_1e_1}_{1}e_2}_{e_2} + w_y\cos\theta\, e_2 + w_y\sin\theta\, \underbrace{e_2e_1e_2}_{-e_1}$$

$$= (w_x\cos\theta - w_y\sin\theta)e_1 + (w_x\sin\theta + w_y\cos\theta)e_2$$

$$(= R_\theta w)$$

multiplying by (uv) on the right rotates w by θ

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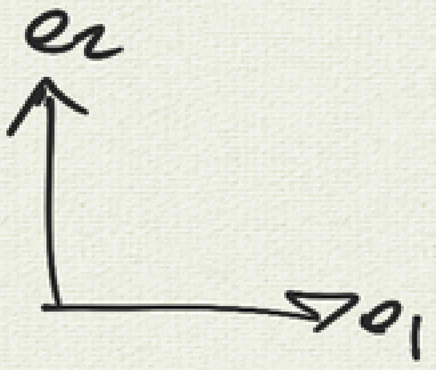
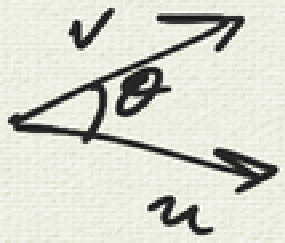
uv rotor



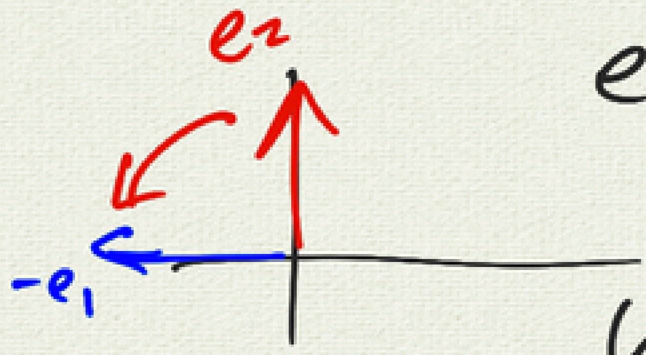


e_1e_2 = rotation by $\pi/2$

uv = rotation by θ

$e_1(e_1e_2) = e_2$ (e_1 rotated $\pi/2$ is e_2)

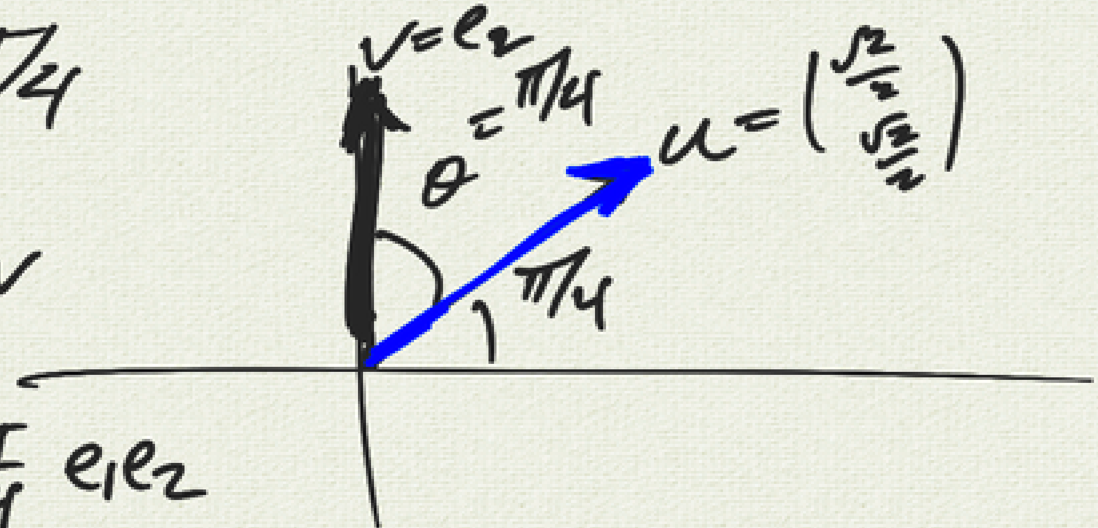
$e_2(e_1e_2) = -e_1$



$$(e_1e_2)^2 = e_1e_2\underbrace{e_1e_2}$$
$$= e_1e_2(-e_2e_1)$$
$$= -e_1\underbrace{e_2e_2}e_1$$
$$= -1$$

$i = e_1e_2$

rotation by $\pi/4$



$uv = u \cdot v + u \wedge v$

$= \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\, e_1 e_2$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e_1 e_2$

<u>rotate u:</u> $\quad u(uv) \leftarrow$ multiply by rotor on right

$= \left(\frac{\sqrt{2}}{2} e_1 + \frac{\sqrt{2}}{2} e_2\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e_1 e_2\right)$

$= \frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{2} e_2 - \frac{1}{2} e_1$

$= e_2$