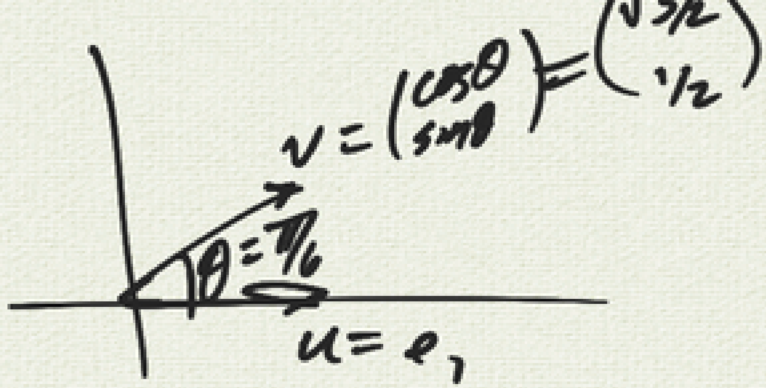
(2c) $(vu)v = \underline{v(uv)}$

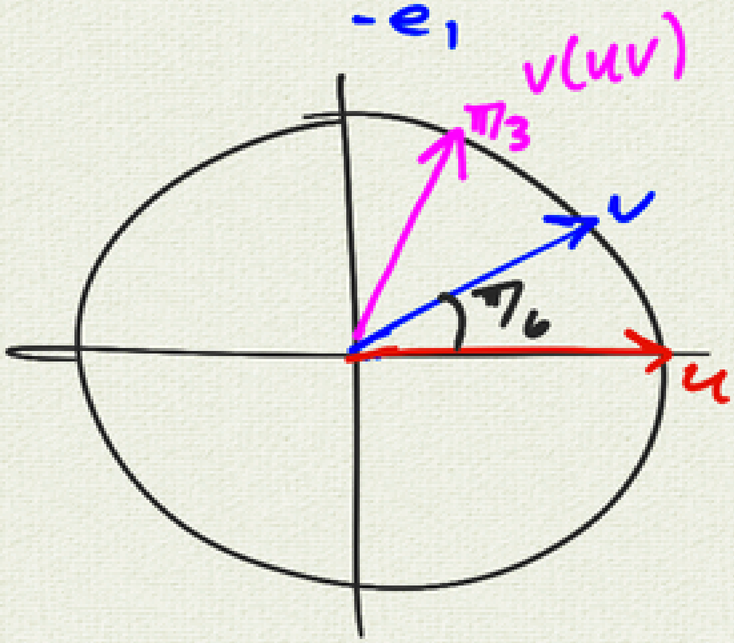


$v(\underbrace{uv})$

$$uv = u \cdot v + u \wedge v$$
$$= \boxed{\underbrace{\frac{\sqrt{3}}{2}}_{\text{scalar}} + \underbrace{\frac{1}{2}e_1e_2}_{\text{bivector}}}$$

$$\underbrace{v(uv)} = \left(\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}e_1e_2\right)$$
$$= \frac{3}{4}e_1 + \frac{\sqrt{3}}{4}\underbrace{e_1e_1}_{1}e_2 + \frac{\sqrt{3}}{4}e_2 + \frac{1}{4}\underbrace{e_2e_1e_2}_{-e_1}$$
$$= \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$$

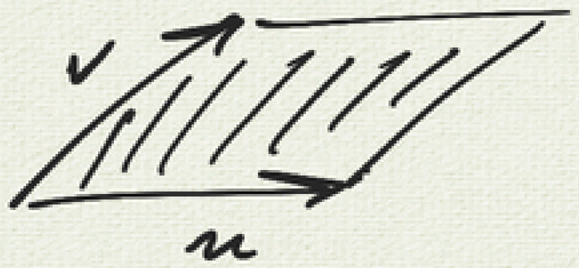
v rotated by $\pi/6$



uv = "rotation by $\pi/6$"
rotor

6.3 Wedge Product in $\mathbb{R}^3$

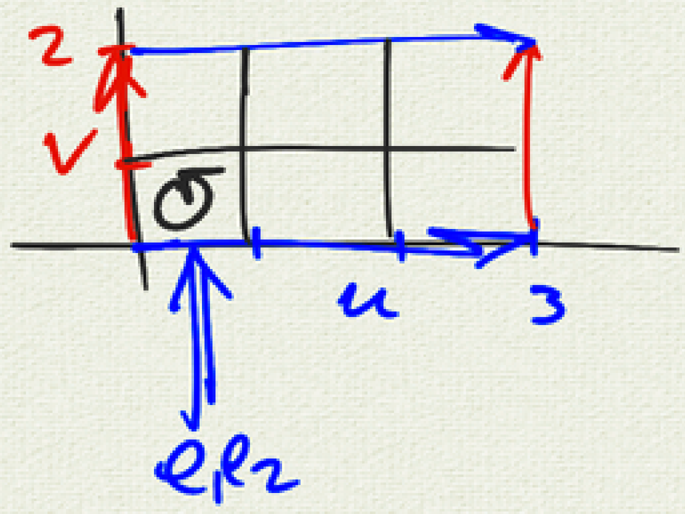
$u \wedge v = $ directed area of $\square$



example: $3e_1 \wedge 2e_2$

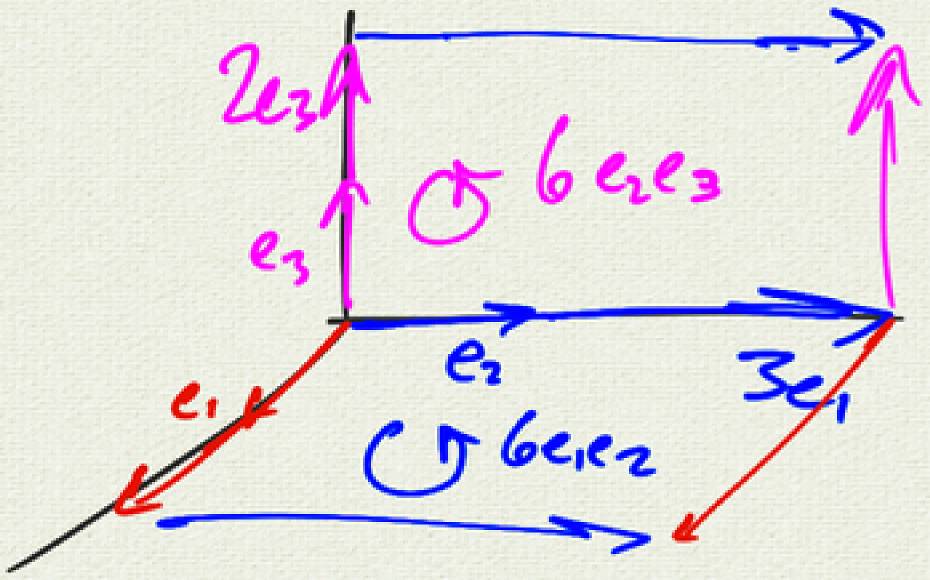
$= 6\, e_1 \wedge e_2$

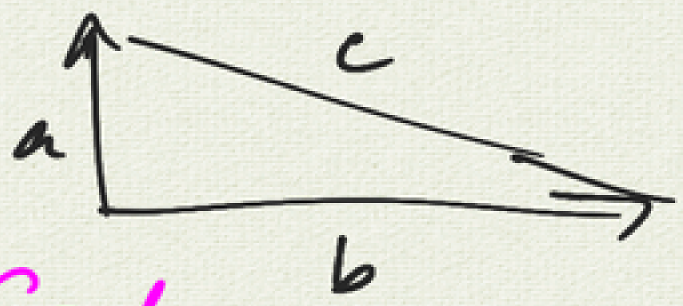
$= 6\, e_1 e_2$

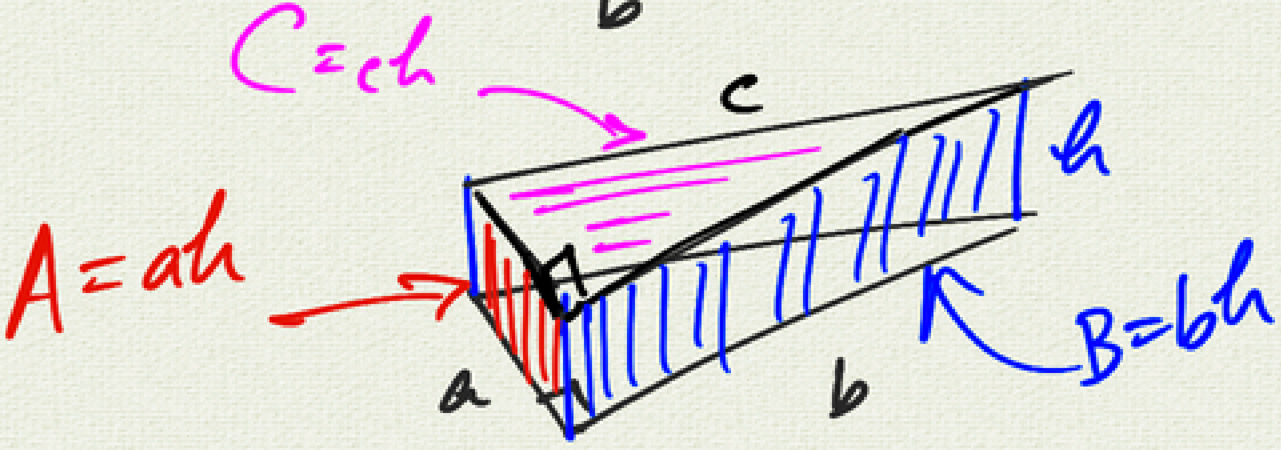


$3e_2 \wedge 2e_3$

$= 6\, e_2 e_3$







$a^2 + b^2 = c^2$

$a^2h^2 + b^2h^2 = c^2h^2$

$(ah)^2 + (bh)^2 = (ch)^2$

$A^2 + B^2 = C^2$

Pythagorean Thm for area

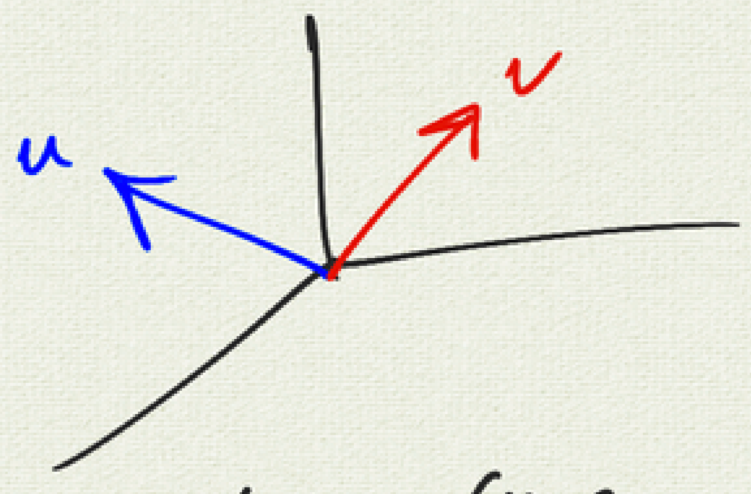
$\Rightarrow$ 3 components for area:

$e_1e_2 \quad e_3e_1 \quad e_2e_3$

general bivector

$a\, e_1e_2 + b\, e_3e_1 + c\, e_2e_3$

$\Rightarrow$ area $= \sqrt{a^2 + b^2 + c^2}$



$$u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = u_x e_1 + u_y e_2 + u_z e_3$$

$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x e_1 + v_y e_2 + v_z e_3$$

$$u \wedge v = (u_x e_1 + u_y e_2 + u_z e_3) \wedge (v_x e_1 + v_y e_2 + v_z e_3)$$

$e_1 \wedge e_1 = 0$
$e_2 \wedge e_2 = 0$
$e_3 \wedge e_3 = 0$

$$= e_1 e_2 (u_x v_y - u_y v_x) + e_3 e_1 (u_z v_x - u_x v_z) + e_2 e_3 (u_y v_z - u_z v_y)$$

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = e_1 \underbrace{\begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}}_{u_y v_z - u_z v_y} - e_2 \underbrace{\begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}}_{u_x v_z - u_z v_x} + e_3 \underbrace{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}_{u_x v_y - v_x u_y}$$

$\Rightarrow |u \times v| =$ area of ▱

$|u \times v| = |u \wedge v| =$ area of ▱

---

let $w = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = w_x e_1 + w_y e_2 + w_z e_3$

$\Rightarrow u \wedge v \wedge w = (u \wedge v) \wedge w$

$$= \left[e_1 e_2 (u_x v_y - u_y v_x) + e_3 e_1 (u_z v_x - u_x v_z) + e_2 e_3 (u_y v_z - u_z v_y) \right] \wedge (w_x e_1 + w_y e_2 + w_z e_3)$$

$u \wedge u = 0$
$e_1 \wedge e_1 = 0$

$$= e_1 e_2 e_3 \left[w_z (u_x v_y - u_y v_x) \right] + w_y (u_z v_x - u_x v_z) e_3 e_1 e_2$$

$- e_1 e_3 e_2$
$+ e_1 e_2 e_3$

$$+ w_x (u_y v_z - u_z v_y) \underbrace{e_2 e_3 e_1}_{e_1 e_2 e_3}$$

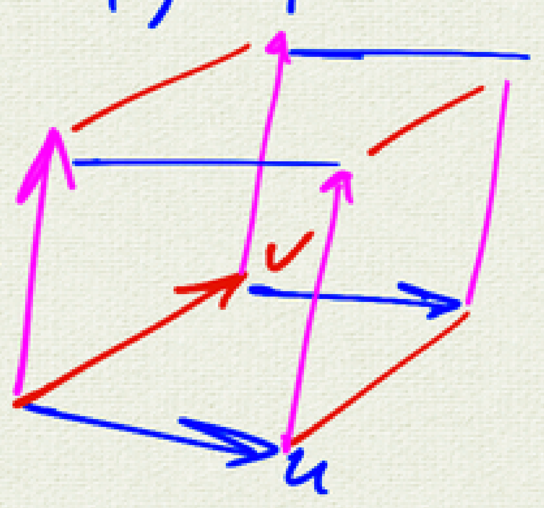
$$= e_1 e_2 e_3 \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$w_x \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}$

$$\boxed{u \wedge v \wedge w = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} e_1 e_2 e_3}$$

trivector

volume of parallelopiped



"triple scalar product"

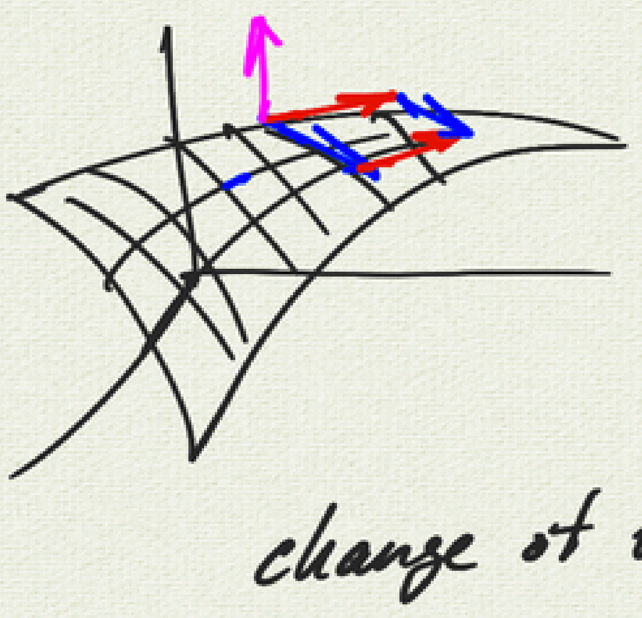
$$(u \times v) \cdot w = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$\mathbb{R}^2$: 1 (scalars); e_1 e_2 (vectors); $e_1 e_2$ (bivector)

$\mathbb{R}^3$: 1 (scalars); $\underbrace{e_1 \ e_2 \ e_3}_{\text{vectors}}$ (length); $\underbrace{e_1 e_2 \ e_2 e_3 \ e_3 e_1}_{\text{bivectors}}$ (area); $e_1 e_2 e_3$ trivector (volume)

---

recall: surface area $\bar{r}(u,v)$



$$\text{area} = \iint |\bar{r}_u \times \bar{r}_v| \, du \, dv = \iint |\bar{r}_u \wedge \bar{r}_v| \, du \, dv$$

change of variables:

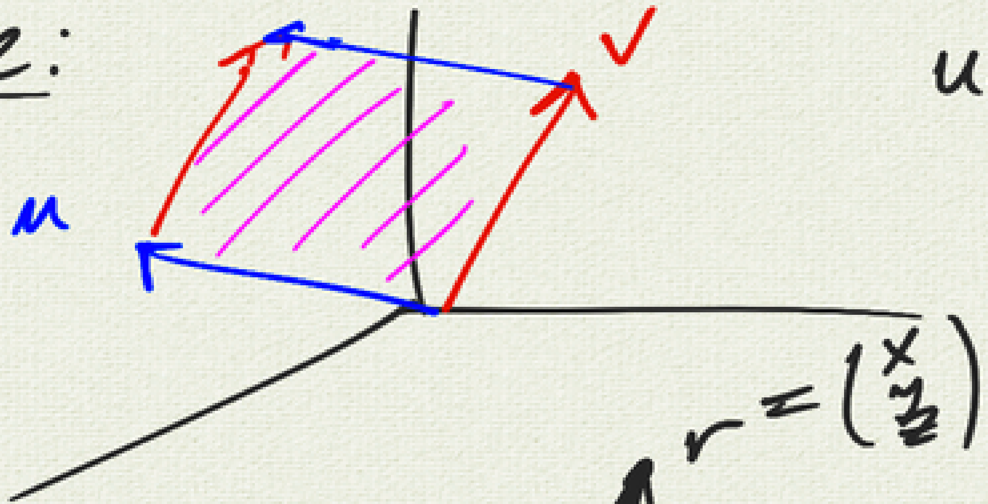
$x = r\cos\theta \quad \Rightarrow \quad dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta$

$y = r\sin\theta \qquad\quad dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta$

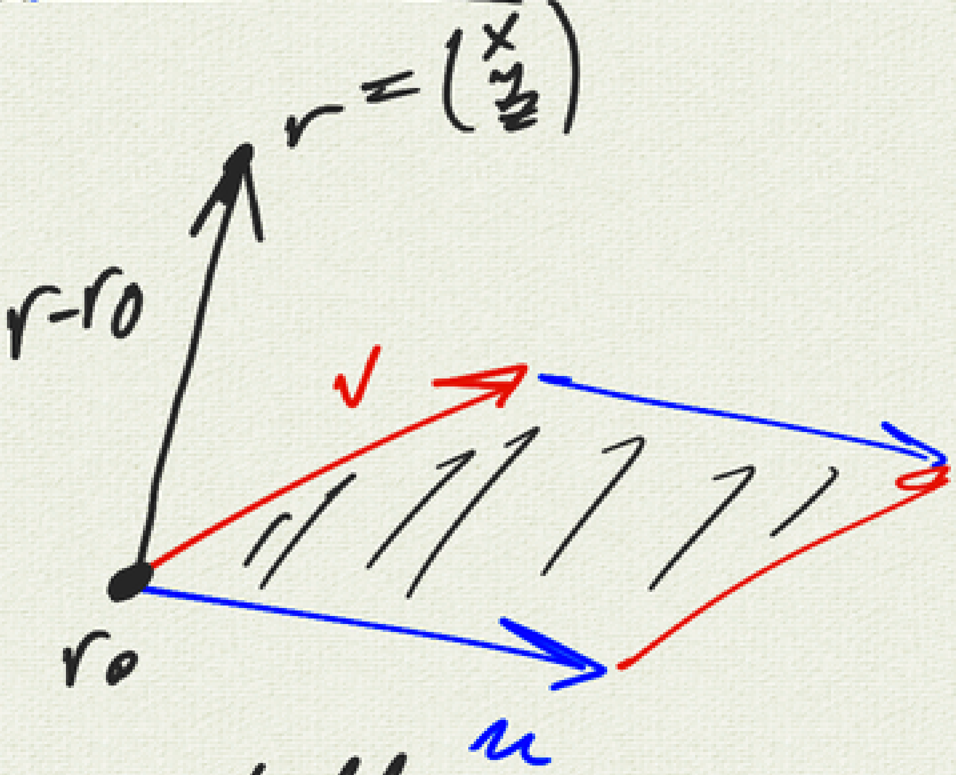
$$\Rightarrow \iint dx\,dy = \iint dx \wedge dy = \iint \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr \wedge d\theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

plane:



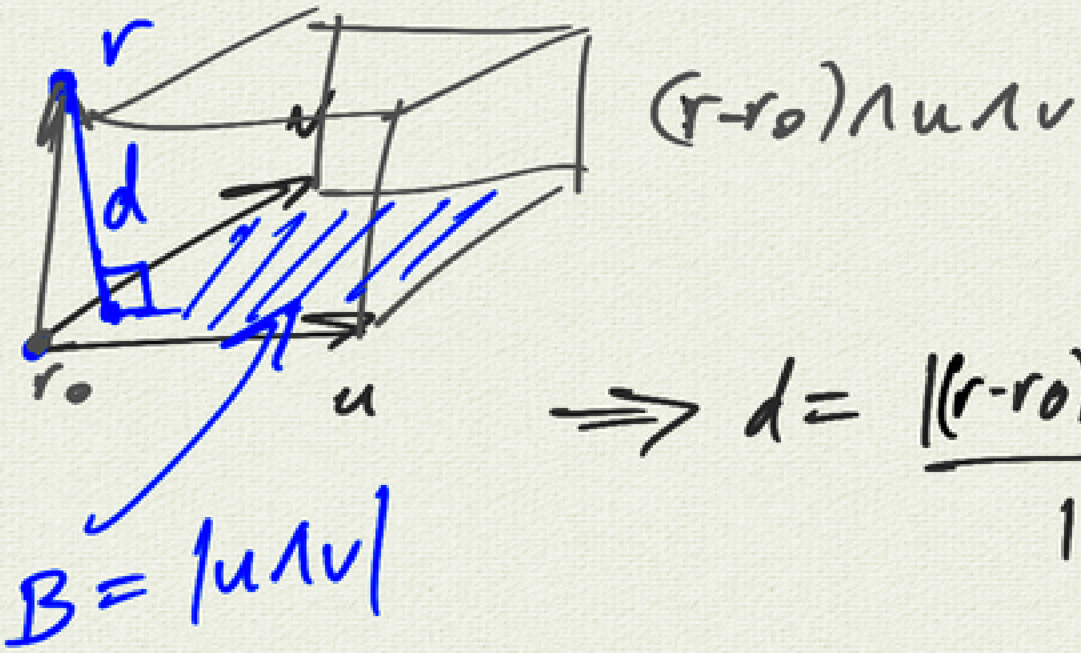
$u \wedge v$ determines a plane



$(r-r_0)\wedge u \wedge v$ = directed volume

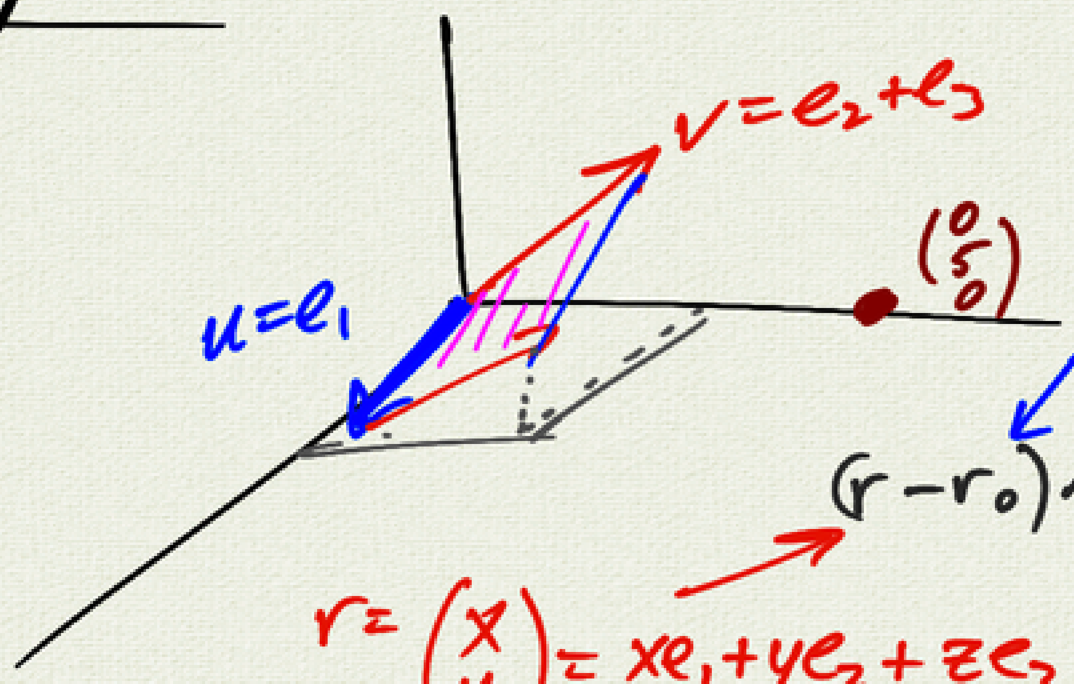
$\Longrightarrow$ r is in the uv plane

$\Longleftrightarrow$ $\boxed{(r-r_0)\wedge u \wedge v = 0}$



$\Longrightarrow$ $d = \dfrac{|(r-r_0)\wedge u \wedge v|}{|u \wedge v|}$

example:



find equation of plane

$u = e_1$

$v = e_2 + e_3$

take $r_0 = 0$

$$(r - r_0) \wedge u \wedge v = 0$$

$$r = \begin{pmatrix}x\\y\\z\end{pmatrix} = xe_1 + ye_2 + ze_3$$

$$(xe_1 + ye_2 + ze_3) \wedge e_1 \wedge (e_2 + e_3) = 0$$

$$\underset{-ye_1e_2e_3}{ye_2e_1e_3} + \underset{e_1e_2e_3}{ze_3e_1e_2} = 0$$

$$(-y + z)e_1e_2e_3 = 0$$

$$z = y$$

---

calculate distance from plane to $\begin{pmatrix}0\\5\\0\end{pmatrix} = 5e_2$

$$d = \frac{|(r - r_0) \wedge u \wedge v|}{|u \wedge v|}$$

$r_0 = 0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$= \frac{|5e_2 \wedge e_1 \wedge (e_2 + e_3)|}{|e_1 \wedge (e_2 + e_3)|}$$

$$= \frac{5}{\sqrt{2}}$$

$e_1 \wedge (e_2 + e_3)$

$= e_1e_2 + e_1e_3$

$= 1e_1e_2 + 1e_1e_3 + 0e_2e_3$

$\Rightarrow |e_1 \wedge (e_1 + e_3)| = \sqrt{2}$