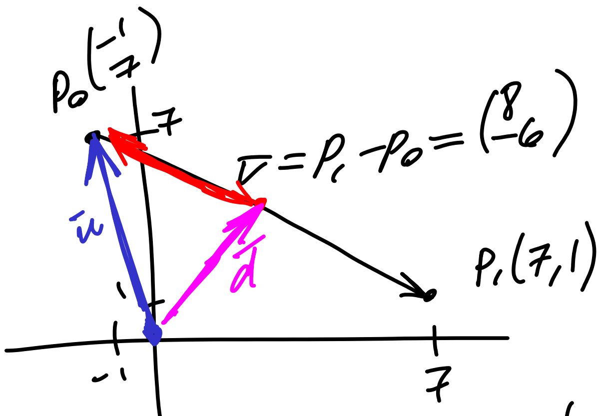
Unit 1 Test
MultiV 2021-22 / Dr. Kessner

Calculator for final approximations only. Have fun!

1. Let $P_0 = (-1, 7)$ and $P_1 = (7, 1)$ be points in the plane. Find parametric equations for the line through these two points. Also find the distance from the line to the origin.



line:

$$\bar{r}(t) = \bar{p}_0 + t\bar{v}$$
$$= \begin{pmatrix} -1 \\ 7 \end{pmatrix} + t\begin{pmatrix} 8 \\ -6 \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

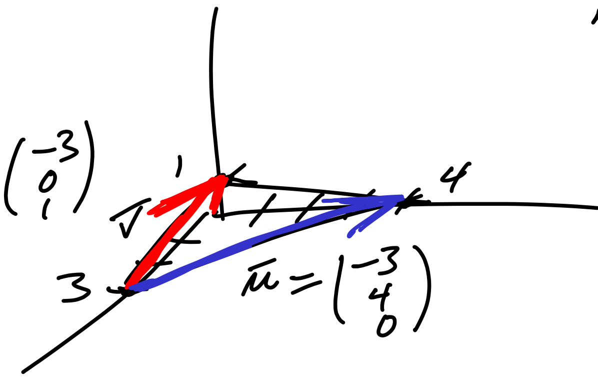
$$|\mathrm{proj}_v(u)| = \bar{u} \cdot \frac{\bar{v}}{|\bar{v}|}$$
$$= \begin{pmatrix} -1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} \frac{1}{5}$$
$$= (-4 - 21)\frac{1}{5}$$
$$= -5$$

$$\mathrm{proj}_v(u) = -5\frac{\bar{v}}{|\bar{v}|}$$
$$= \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

rejection $\bar{d} = \bar{u} - \mathrm{proj}_v(u)$
$$= \begin{pmatrix} -1 \\ 7 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

distance $d = |\bar{d}| = 5$

2. Find the equation of the plane through the points $(3, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 1)$ using vectors. Find the distance from the plane to the point $(1, 7, 13)$.



$$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & 0 \\ -3 & 0 & 1 \end{vmatrix}$$

$$= 4\vec{i} + 3\vec{j} + 12\vec{k}$$

point $p_0 = \begin{pmatrix}3\\0\\0\end{pmatrix}$

plane

$$\bar{n} \cdot (\bar{r} - \bar{p}_0) = 0$$

$$\begin{pmatrix}4\\3\\12\end{pmatrix} \cdot \begin{pmatrix}x-3\\y\\z\end{pmatrix} = 0$$

$$4(x-3) + 3y + 12z = 0$$

$$4x + 3y + 12z = 12$$

$$\bar{p}_1 = \begin{pmatrix}1\\7\\13\end{pmatrix}$$

$$\text{distance} = \left| (\bar{p}_1 - \bar{p}_0) \cdot \frac{\bar{n}}{|\bar{n}|} \right|$$

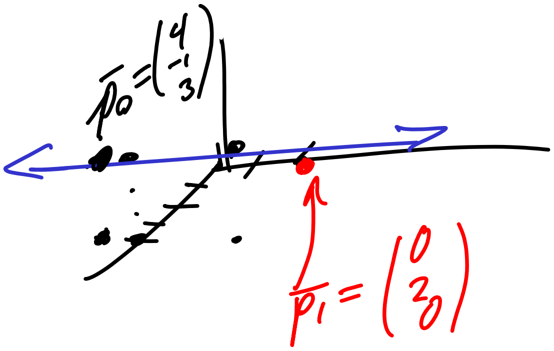
$$= \begin{pmatrix}-2\\7\\13\end{pmatrix} \cdot \begin{pmatrix}4\\3\\12\end{pmatrix} \frac{1}{13}$$

$$= (-8 + 21 + 156) \frac{1}{13}$$

$$= 169/13$$

$$= 13$$

3. Find parametric equations for the line through the points $(4,-1,3)$ and $(4,2,3)$ in $\mathbb{R}^3$. Also find the distance from the line to the point $(0,2,0)$.



line $\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\bar{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\bar{v} = \bar{p}_1 - \bar{p}_0 = \begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix}$$

$$|\mathrm{proj}_{\bar{u}} \bar{v}| = \bar{v} \cdot \frac{\bar{u}}{|\bar{u}|}$$

$$= \begin{pmatrix} -4 \\ +3 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= +3$$

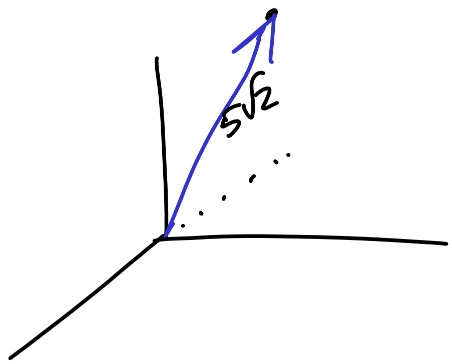
$$\mathrm{proj}\, v = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\bar{d} = \bar{v} - \mathrm{proj}\, \bar{v} = \begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -3 \end{pmatrix}$$

$$|d| = 5$$

4. Find both cylindrical and spherical coordinates for the following points (given in rectangular coordinates):

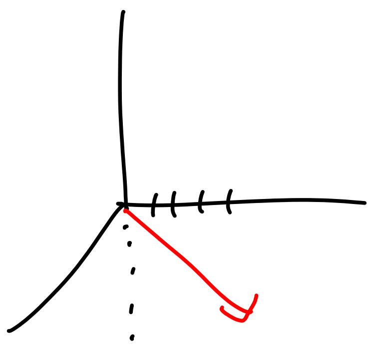
a. $(-5, 0, 5)$



$(r, \theta, z) = (5, \pi, 5)$

$(\rho, \theta, \varphi) = (5\sqrt{2}, \pi, \pi/4)$

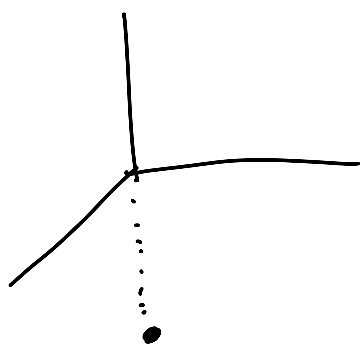
b. $(0, 4, -4)$



$(r, \theta, z) = (4, \frac{\pi}{2}, -4)$

$(\rho, \theta, \varphi) = (4\sqrt{2}, \frac{\pi}{2}, \frac{3\pi}{4})$

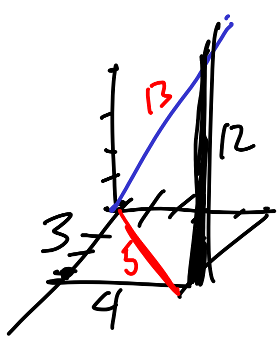
c. $(0, 0, -10)$



$(r, \theta, z) = (0, 0, -10)$

$(\rho, \theta, \varphi) = (10, 0, \pi)$

d. $(3, 4, 12)$



$(r, \theta, z) = (5, \tan^{-1}\frac{4}{3}, 12)$

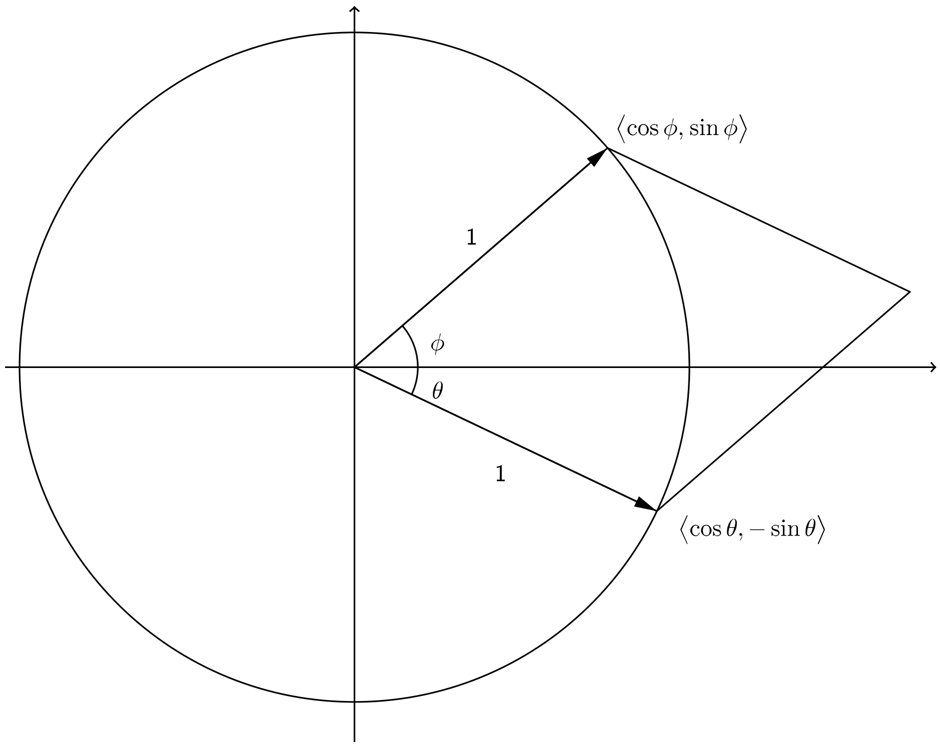
$(\rho, \theta, \varphi) = (13, \tan^{-1}\frac{4}{3}, \sin^{-1}\frac{5}{13})$

Bonus Consider the vector $\vec{v} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$.

Recall that rotation by θ is represented by the matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Rotate vector $\vec{v}$ by angle θ to derive the sum formulas for sin and cos.

Bonus



In the diagram above, θ and ϕ are positive. Recall that the area of the parallelogram determined by two vectors is the determinant (or magnitude of the cross product).

Consider the area of the parallelogram above to derive the sum formula for sine:

$$\sin(\theta + \phi) = \cos\theta \sin\phi + \sin\theta \cos\phi$$