Unit 2 Group Work
MultiV 2021-22 / Dr. Kessner

KEY

No calculator! Have fun!

1. Consider the general helix:

$$\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$$

a. Calculate the velocity, speed, and acceleration of this curve.

$\vec r\,'(t) = \begin{pmatrix}-a\sin t\\ a\cos t\\ b\end{pmatrix}$ $\qquad \vec r\,''(t) = \begin{pmatrix}-a\cos t\\ -a\sin t\\ 0\end{pmatrix}$

b. Calculate the tangent $\mathbf{T}(t)$, normal $\mathbf{N}(t)$, and binormal $\mathbf{B}(t)$ vector functions.

$|\vec r\,'(t)| = \sqrt{a^2+b^2}$

$T = \dfrac{\vec r\,'(t)}{|\vec r\,'(t)|} = \dfrac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}a\sin t\\ a\cos t\\ b\end{pmatrix}$

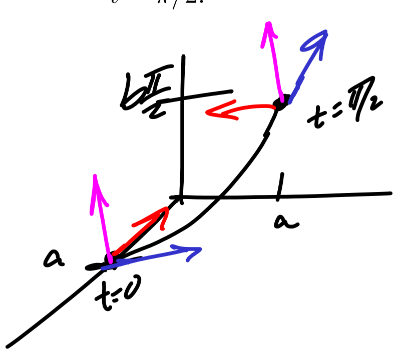
$T'(t) = \dfrac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-a\cos t\\ -a\sin t\\ 0\end{pmatrix}$

$N(t) = \begin{pmatrix}-\cos t\\ -\sin t\\ 0\end{pmatrix}$

$B = T\times N = \dfrac{1}{\sqrt{a^2+b^2}}\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ -a\sin t & a\cos t & b\\ -\cos t & -\sin t & 0\end{vmatrix}$

$= \dfrac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}b\sin t\\ -b\cos t\\ a\end{pmatrix}$

c. Find the equations of the osculating (TN), normal (NB), and rectifying (TB) planes at $t = 0$ and $t = \pi/2$.



$\underline{t=0}$ $(a,0,0)$

$\underline{TN}$ $\vec n = B = \dfrac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}0\\ -b\\ a\end{pmatrix}$

$-by + az = 0$

$\underline{NB}$ $\vec n = T = \begin{pmatrix}0\\ a\\ b\end{pmatrix}\dfrac{1}{\sqrt{a^2+b^2}}$

$ay + bz = 0$

$\underline{TB}$ $\vec n = N = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$

$x = a$

$\underline{t=\pi/2}$ $(0, a, \frac{b\pi}{2})$

$\underline{TN}$ $\vec n = B = \dfrac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}b\\ 0\\ a\end{pmatrix}$

$bx + a(z - \frac{b\pi}{2}) = 0$

$bx + az = \frac{ab\pi}{2}$

$\underline{NB}$ $\vec n = T = \dfrac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-a\\ 0\\ b\end{pmatrix}$

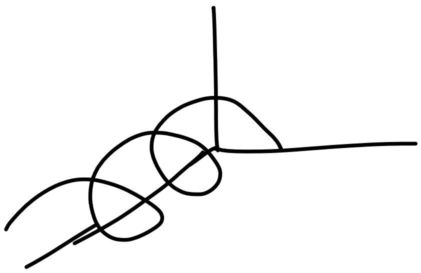
$-ax + b(z - b\pi/2) = 0$

$-ax + bz = b^2\pi/2$

$\underline{TB}$ $\vec n = N = \begin{pmatrix}0\\ -1\\ 0\end{pmatrix}$

$y = a$

2. a. Parametrize the general helix as in #1, but this time oriented along the x-axis. (As in #1, let a be the radius and b the speed along the axis).



$$\vec{r}(t) = \begin{pmatrix} bt \\ a\cos t \\ a\sin t \end{pmatrix} \qquad \vec{r}\,'(t) = \begin{pmatrix} b \\ -a\sin t \\ a\cos t \end{pmatrix}$$

$$|\vec{r}\,'(t)| = \sqrt{a^2+b^2}$$

b. Calculate the arc length of the curve as a function of t, starting from $t = 0$.

$$s(t) = \int_0^t |\vec{r}\,'(t)|\,dt = \sqrt{a^2+b^2}\,t$$

c. Re-parametrize the curve by arc length.

$$t = \frac{1}{\sqrt{a^2+b^2}}\,s$$

$$\vec{r}(s) = \begin{pmatrix} bs/\sqrt{a^2+b^2} \\ a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \end{pmatrix}$$

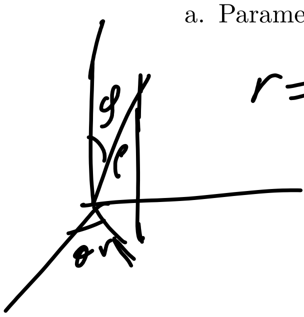
3. Consider the curve given by the spherical equations:

$$\rho = 4$$

$$\theta = \frac{\pi}{4}$$

$$\phi = t$$

a. Parametrize the curve in standard rectangular coordinates $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. What is this curve?



$r = \rho \sin \varphi = 4 \sin t$

$z = \rho \cos \varphi = 4 \cos t$

$x = r \cos \theta$

$= 4 \sin t \cos \pi/4$

$= 2\sqrt{2} \sin t$

$y = r \sin \theta$

$= 4 \sin t \cdot \frac{\sqrt{2}}{2}$

$= 2\sqrt{2} \sin t$

$$\vec{r}(t) = \begin{pmatrix} 2\sqrt{2} \sin t \\ 2\sqrt{2} \sin t \\ 4 \cos t \end{pmatrix} \qquad \vec{r}'(t) = \begin{pmatrix} 2\sqrt{2} \cos t \\ 2\sqrt{2} \cos t \\ -4 \sin t \end{pmatrix}$$

$|\vec{r}'(t)| = 4$

b. Find the arc length of the curve as a function of t, and re-parametrize by arc length.

$$\vec{r}(s) = \begin{pmatrix} 2\sqrt{2} \sin(s/4) \\ 2\sqrt{2} \sin(s/4) \\ 4 \cos(s/4) \end{pmatrix}$$

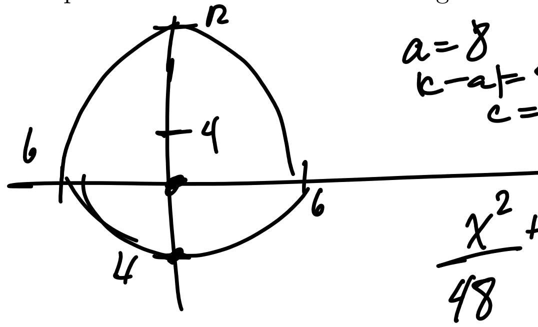
4. a. Consider the following function defined in polar coordinates:

$$r = \frac{12}{2 - \sin\theta}$$

graph on Desmos to check

Sketch the graph of the curve. Also find the equation of the function in rectangular coordinates.

θ	r
0	6
$\pi/2$	12
π	6
$3\pi/2$	4
2π	6



$a = 8$

$|c - a| = 4$

$c = 4$

$b^2 = a^2 - c^2$

$= 64 - 16$

$= 48$

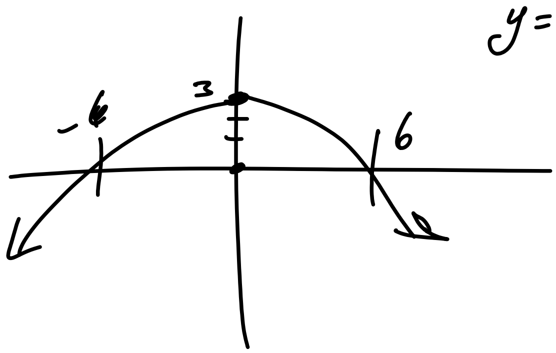
$$\frac{x^2}{48} + \frac{(y-4)^2}{64} = 1$$

b. Consider the following function defined in polar coordinates:

$$r = \frac{12}{2 + 2\sin\theta}$$

Sketch the graph of the curve. Also find the equation of the function in rectangular coordinates.

θ	$r = \frac{12}{2+2\sin\theta}$
0	6
$\pi/2$	3
π	6
$3\pi/2$	undef



$$y = 3 - \frac{1}{12}x^2$$