Unit 2 Test
MultiV 2021-22 / Dr. Kessner

No calculator! Have fun!

1. Consider the curve defined by the vector function

$$\mathbf{r}(t) = \begin{pmatrix} 10\cos t \\ 5\sqrt{2}\sin t \\ 5\sqrt{2}\sin t \end{pmatrix}$$

What is the curve?

a) Find the velocity and speed of this curve.

b) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

c) Find the equations of the osculating (TN), normal (NB), and rectifying (TB) planes at $t = 0$.

2. Consider the curve defined by:

$$\mathbf{r}(t) = \begin{pmatrix} \cos 5t^2 \\ \sin 5t^2 \\ 7 \end{pmatrix}$$

a) Find the velocity and speed of this curve.

b) Find the arc length of the curve $s(t)$, starting from $t = 0$.

c) Reparametrize the curve by arc length. What is the curve?

3. Suppose that a projectile is launched (in 3D, z is up) from a platform 80 ft. above the origin. The initial position $(x_0, y_0, z_0) = (0, 0, 80)$ and initial velocity $(v_x, v_y, v_z) = (30, 40, 64)$.

a) Assume that the acceleration $\mathbf{a}(t) = \langle 0, 0, -32 \rangle$, i.e. gravity acts in the z direction. Find the velocity vector $\mathbf{v}(t)$.

b) Find the position vector $\mathbf{r}(t)$.

c) What is the maximum height (z value) of the projectile?

d) The projectile hits the ground when $z(t) = 0$. When does this happen? What is the (x, y) position when it hits the ground? What is the distance from the origin to the point of impact?

4. a. Consider the following function defined in polar coordinates:

$$r = \frac{6}{1 + 2\cos\theta}$$

Sketch the graph of the curve. Also find the equation of the curve in rectangular coordinates.

b. Consider the following function defined in polar coordinates:

$$r = \frac{6}{2 + \cos\theta}$$

Sketch the graph of the curve. Also find the equation of the curve in rectangular coordinates.

Bonus Suppose $\mathbf{r}(t)$ is a curve on the unit sphere. Prove that the velocity vector is always orthogonal to the position vector.

Bonus Show that the result above implies that $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ are pairwise orthogonal unit vectors for any curve $\mathbf{r}(t)$, for all t. (In other words, $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ form what is known as an *orthonormal basis* for $\mathbb{R}^3$.)

Bonus Assume Newton's second law: $\mathbf{F} = m\mathbf{a}$. Suppose that an object is moving with its position given by $\mathbf{r}(t)$ and velocity by $\mathbf{v}(t)$. Also suppose that the only force acting on the object is a central force (in other words, the force is always directed toward the origin). Show that $\mathbf{r}(t) \times \mathbf{v}(t)$ is constant. Why does this imply that the motion is constrained to a plane?

Bonus Suppose you are given a point (focus) and a line (directrix). Recall the alternate geometric definition of an ellipse: every point on the ellipse satisfies

$$\frac{\text{distance to focus}}{\text{distance to directrix}} = e < 1$$

Let d be the distance between the focus and directrix. Derive the polar equation for the ellipse.