Unit 2 Test
MultiV 2021-22 / Dr. Kessner

KEY

No calculator! Have fun!

1. Consider the curve defined by the vector function

$$\mathbf{r}(t) = \begin{pmatrix} 10\cos t \\ 5\sqrt{2}\sin t \\ 5\sqrt{2}\sin t \end{pmatrix}$$

What is the curve?

a) Find the velocity and speed of this curve.

$$r'(t) = \begin{pmatrix} -10\sin t \\ 5\sqrt{2}\cos t \\ 5\sqrt{2}\cos t \end{pmatrix} \qquad |r'(t)| = 10$$

b) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

$$T(t) = \frac{1}{10} r'(t) = \begin{pmatrix} -\sin t \\ \sqrt{2}/2 \cos t \\ \sqrt{2}/2 \cos t \end{pmatrix} \qquad B(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \frac{\sqrt{2}}{2}\cos t & \frac{\sqrt{2}}{2}\cos t \\ -\cos t & -\frac{\sqrt{2}}{2}\sin t & -\frac{\sqrt{2}}{2}\sin t \end{vmatrix}$$

$$T'(t) = \begin{pmatrix} -\cos t \\ -\frac{\sqrt{2}}{2}\sin t \\ -\frac{\sqrt{2}}{2}\sin t \end{pmatrix} = N(t) \qquad = \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$

$$|T'(t)| = 1$$

c) Find the equations of the osculating (TN), normal (NB), and rectifying (TB) planes at $t = 0$.

$$r(0) = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{TN}$ $\quad n = B = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$-y + z = 0$

$z = y$

$\underline{NB}$

$n = T = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$y + z = 0$

$\underline{TB}$ $\quad n = N = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$x = 10$

2. Consider the curve defined by:

$$\mathbf{r}(t) = \begin{pmatrix} \cos 5t^2 \\ \sin 5t^2 \\ 7 \end{pmatrix}$$

a) Find the velocity and speed of this curve.

$$\vec{r}\,'(t) = \begin{pmatrix} -10t \sin 5t^2 \\ 10t \cos 5t^2 \\ 0 \end{pmatrix}$$

$$|\vec{r}\,'(t)| = 10t$$

b) Find the arc length of the curve $s(t)$, starting from $t = 0$.

$$s(t) = \int_0^t |\vec{r}\,'(t)|\,dt$$

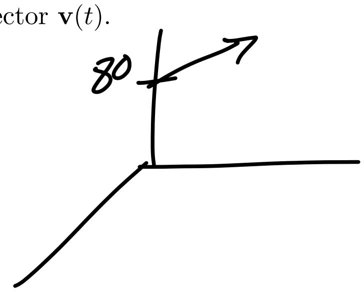
$$= \int_0^t 10t\,dt = 5t^2$$

c) Reparametrize the curve by arc length. What is the curve?

$$\vec{r}(t) = \begin{pmatrix} \cos s \\ \sin s \\ 7 \end{pmatrix}$$

3. Suppose that a projectile is launched (in 3D, z is up) from a platform 80 ft. above the origin. The initial position $(x_0, y_0, z_0) = (0, 0, 80)$ and initial velocity $(v_x, v_y, v_z) = (30, 40, 64)$.

a) Assume that the acceleration $\mathbf{a}(t) = \langle 0, 0, -32 \rangle$, i.e. gravity acts in the z direction. Find the velocity vector $\mathbf{v}(t)$.



$$\bar{v}(t) = \begin{pmatrix} v_x \\ v_y \\ v_z - 32t \end{pmatrix} = \begin{pmatrix} 30 \\ 40 \\ 64 - 32t \end{pmatrix}$$

b) Find the position vector $\mathbf{r}(t)$.

$$\bar{r}(t) = \begin{pmatrix} 30t \\ 40t \\ 80 + 64t - 16t^2 \end{pmatrix}$$

c) What is the maximum height (z value) of the projectile?

$$z(t) = -16t^2 + 64t + 80$$
$$= -16(t^2 - 4t - 5)$$
$$= -16(t-5)(t+1)$$

$$z'(t) = -32t + 64$$
$z'(t) = 0$ when $t = 2$
$$z(2) = 144$$

d) The projectile hits the ground when $z(t) = 0$. When does this happen? What is the (x, y) position when it hits the ground? What is the distance from the origin to the point of impact?

$z(t) = 0$ at $t = 5$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 150 \\ 200 \end{pmatrix}$$

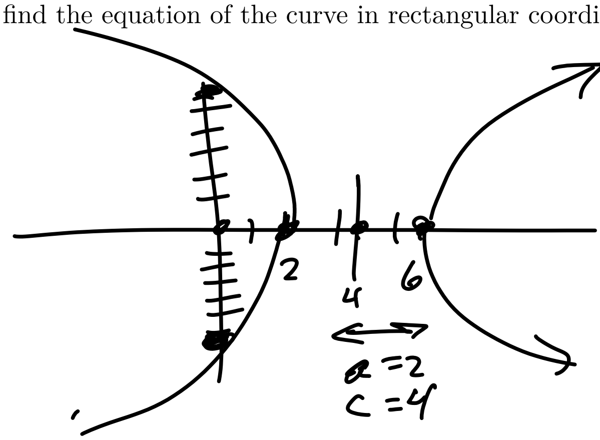
$$r = 250$$

4. a. Consider the following function defined in polar coordinates:

$$r = \frac{6}{1 + 2\cos\theta}$$

Sketch the graph of the curve. Also find the equation of the curve in rectangular coordinates.

θ	r
0	2
$\pi/2$	6
π	-6
$3\pi/2$	6



$e=2$ hyperbola

$a=2$
$c=4$

$b^2=16-4=12$

center $(4,0)$

$$\frac{(x-4)^2}{4} - \frac{y^2}{12} = 1$$

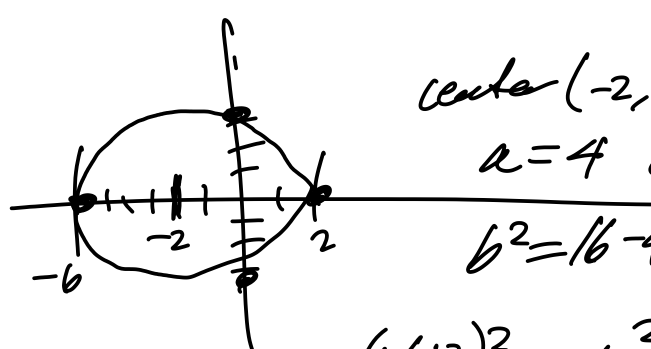
b. Consider the following function defined in polar coordinates:

$$r = \frac{6}{2 + \cos\theta}$$

Sketch the graph of the curve. Also find the equation of the curve in rectangular coordinates.

$e = 1/2$ ellipse

θ	r
0	2
$\pi/2$	3
π	6
$3\pi/2$	3



center $(-2,0)$

$a=4$ $c=2$

$b^2=16-4=12$

$$\frac{(x+2)^2}{16} + \frac{y^2}{12} = 1$$

Bonus Suppose $\mathbf{r}(t)$ is a curve on the unit sphere. Prove that the velocity vector is always orthogonal to the position vector.

Bonus Show that the result above implies that $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ are pairwise orthogonal unit vectors for any curve $\mathbf{r}(t)$, for all t. (In other words, $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ form what is known as an *orthonormal basis* for $\mathbb{R}^3$.)

Bonus Assume Newton's second law: $\mathbf{F} = m\mathbf{a}$. Suppose that an object is moving with its position given by $\mathbf{r}(t)$ and velocity by $\mathbf{v}(t)$. Also suppose that the only force acting on the object is a central force (in other words, the force is always directed toward the origin). Show that $\mathbf{r}(t) \times \mathbf{v}(t)$ is constant. Why does this imply that the motion is constrained to a plane?

Bonus Suppose you are given a point (focus) and a line (directrix). Recall the alternate geometric definition of an ellipse: every point on the ellipse satisfies

$$\frac{\text{distance to focus}}{\text{distance to directrix}} = e < 1$$

Let d be the distance between the focus and directrix. Derive the polar equation for the ellipse.