KEY

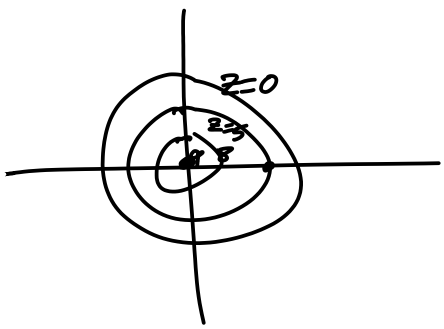
Unit 3 Test
MultiV 2021-22 / Dr. Kessner

Name:

Pledge:

1. Let $z = f(x, y) = 9 - x^2 - y^2$.

a. Draw level sets of f in the xy-plane. What is this surface?



b. At the following points, find the direction of maximum change $\mathbf{u}$ and the directional derivative in that direction $D_{\mathbf{u}}(f)$: $(\pm 3, 0), (0, \pm 3), (0, 0)$.

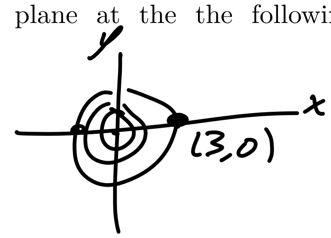
$$f(x,y) = 9 - x^2 - y^2$$

$$\nabla f = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

$\nabla f(3,0) = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$ $\quad \nabla f(-3,0) = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ $\quad \nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \nabla f(0,3) = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$ $\quad \nabla f(0,-3) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

$|\nabla f| = 6$ $\quad |\nabla f| = 6$ $\quad |\nabla f| = 0$ $\quad |\nabla f| = 6$ $\quad |\nabla f| = 6$

c. On the same surface, find the equation of the tangent plane at the the following points: $(\pm 3, 0), (0, \pm 3), (0, 0)$.



$(3,0)$: $\nabla f = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$

$z = 0 - 6(x-3) + 0(y-0)$

$= -6x + 18$

$(-3,0)$ $z = 6(x+3) = 6x + 18$

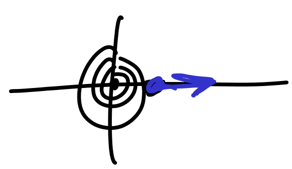
$(0,0)$: $z = 9$

$(0,3)$: $z = -6(y-3)$

$= -6y + 18$

$(0,-3)$: $z = 6y + 18$

d. Consider the curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 3 + t, 0 \rangle$, and the composite function $f(t) = f(x(t), y(t))$. Find $f'(t)$ in two different ways: 1) standard differentiation, and 2) multivariable chain rule.



$r(t) = \begin{pmatrix} 3+t \\ 0 \end{pmatrix}$ $\quad f(x,y) = 9 - x^2 - y^2$

$f(r(t)) = 9 - (3+t)^2 - 0^2$

$= 9 - (9 + 6t + t^2)$

$= -t^2 - 6t$

$f'(t) = -2t - 6$

$f'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$= (-2x)(1) + (-2y)(0)$

$= -2(3+t)$

$= -6 - 2t$

e. Consider the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle 3\cos t, 3\sin t\rangle$, and the composite function $f(t) = f(x(t), y(t))$. Find $f'(t)$ in two different ways: 1) standard differentiation, and 2) multivariable chain rule.

$f(x,y) = 9 - x^2 - y^2$

$f(\bar{r}(t)) = 0 \Rightarrow f'(t) = 0$

$\bar{r}(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \end{pmatrix}$

$\bar{r}'(t) = \begin{pmatrix} -3\sin t \\ 3\cos t \end{pmatrix}$

$f'(t) = (-2x)(-3\sin t) + (-2y)(3\cos t)$

$= +18\sin t\cos t - 18\sin t\cos t$

$= 0$

f. Consider the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle (3+t)\cos t, (3+t)\sin t\rangle$, and the composite function $f(t) = f(x(t), y(t))$. Find $f'(t)$ in two different ways: 1) standard differentiation, and 2) multivariable chain rule.

$f(\bar{r}(t)) = 9 - (3+t)^2\cos^2 t - (3+t)^2\sin^2 t$

$= 9 - (3+t)^2$

$= 9 - (9 + 6t + t^2)$

$= -6t - t^2$

$f'(t) = -6 - 2t$

$\bar{r}(t) = \begin{pmatrix} (3+t)\cos t \\ (3+t)\sin t \end{pmatrix}$

$\bar{r}'(t) = \begin{pmatrix} \cos t - (3+t)\sin t \\ \sin t + (3+t)\cos t \end{pmatrix}$

$\nabla f = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$f'(t) = \nabla f \cdot \bar{r}'(t)$

$= (-2x)(\cos t - (3+t)\sin t) - 2y(\sin t + (3+t)\cos t)$

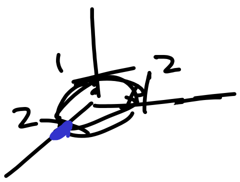
$= -2((3+t)\cos t)(\cos t \underline{- (3+t)\sin t}) - 2(3+t)\sin t(\sin t \underline{+ (3+t)\cos t})$

$= -2(3+t)$

$= -6 - 2t$

2. Let $F(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{4} + z^2$ and consider the level surface $F(x,y,z) = 1$. What is this surface?

a. For each of the following points, verify that it is on the surface, find the direction of maximum change $\mathbf{u}$, and the directional derivative in that direction $D_{\mathbf{u}}(F)$: $(\pm 2,0,0), (0,\pm 2,0), (0,0,\pm 1)$.



$\nabla F = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \\ 2z \end{pmatrix}$

$(2,0,0)$: $\nabla F = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$(-2,0,0)$: $\nabla F = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$(0,2,0)$ $\nabla F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$(0,-2,0)$ $\nabla F = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$(0,0,1)$ $\nabla F = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$(0,0,-1)$ $\nabla F = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$

b. At each of the following points, find the equation of the tangent plane: $(\pm 2,0,0), (0,\pm 2,0), (0,0,\pm 1)$.

$(2,0,0)$ $\nabla F \cdot (\bar{r} - \bar{r}_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y-0 \\ z-0 \end{pmatrix} = 0$

$x = 2$

$(-2,0,0)$ $x = -2$

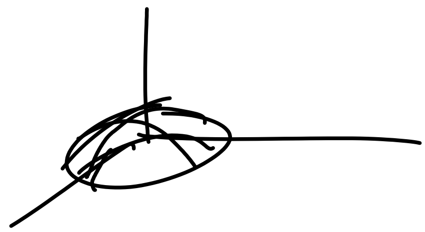
$(0,2,0)$ $y = 2$

$(0,-2,0)$ $y = -2$

$(0,0,1)$ $z = 1$

$(0,0,-1)$ $z = -1$

c. Consider the curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle 2\cos t, 2\sin t, 0\rangle$, and the composite function $F(t) = F(x(t), y(t), z(t))$. Find $F'(0)$ in two different ways: 1) standard differentiation, and 2) multivariable chain rule.



$$F(\bar{r}(t)) = \frac{(2\cos t)^2}{4} + \frac{(2\sin t)^2}{4} + 0^2$$

$$= 1$$

$$F'(t) = 0$$

$$F'(t) = \nabla F \cdot \bar{r}'(t)$$

$$= \cos t(-2\sin t) + \sin t(2\cos t)$$

$$= 0$$

$$\bar{r}'(t) = \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix}$$

$$\nabla F = \begin{pmatrix} x/2 \\ y/2 \\ 2z \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$$

d. Consider the curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle 2 + t, 0, 0\rangle$, and the composite function $F(t) = F(x(t), y(t), z(t))$. Find $F'(0)$ in two different ways: 1) standard differentiation, and 2) multivariable chain rule.

$$F(\bar{r}(t)) = \frac{(2+t)^2}{4} + 0^2 + 0^2$$

$$\Rightarrow F'(t) = \frac{1}{4} \cdot 2(2+t)$$

$$= \frac{1}{2}(2+t)$$

$$= 1 + t/2$$

$$F'(t) = \nabla F \cdot \bar{r}'(t)$$

$$= 1 + t/2$$

$$\nabla F = \begin{pmatrix} x/2 \\ y/2 \\ 2z \end{pmatrix} = \begin{pmatrix} 1 + t/2 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{r}'(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Bonus Suppose you have a surface $z = f(x, y)$. Show that you can represent the surface as a level surface of some function F and that the tangent plane $z = z_0 + f_x \Delta x + f_y \Delta y$ is the same as the plane given by the point (x_0, y_0, z_0) and the normal ∇F.

Bonus Derive the double angle formulas for sin and cos by rotating by θ twice, using the rotation matrix R_θ.

Bonus Suppose that $F(x, y, z) = \text{const}$ is a level surface, and $\mathbf{r}(t)$ is a curve on the surface. Show that at any point along the curve, the tangent vector of the curve is perpendicular to the gradient of the function. (Equivalently, the tangent vector of the curve lies in the tangent plane to the surface.)

Bonus Derive the multivariable chain rule. *Hint: composition of linear transformations.*

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$