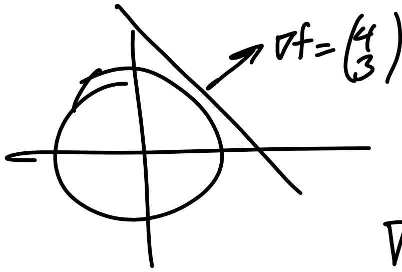
Unit 4 Test
MultiV 2021-22 / Dr. Kessner

KEY

No calculator! Remember to verify local minima with a 2nd derivative test where appropriate! Have fun!

1. a) Find the points where $f(x, y) = 4x + 3y$ is maximized and minimized on the unit circle.



$g(x,y) = x^2 + y^2 = 1$

$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$\nabla f = \lambda \nabla g$

$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$4 = 2\lambda x$

$3 = 2\lambda y$

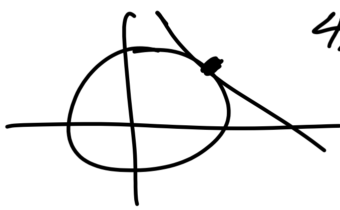
$\frac{4}{x} = \frac{3}{y}$

$4y = 3x$

$x^2 + \left(\frac{3}{4}\right)x^2 = 1$

$x^2 + y^2 = 1 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$

b) For each of these points, find the equation of the line tangent to the unit circle at that point.



$4x + 3y = 1$

2. Find the dimensions that maximize the volume of a cylinder with surface area 600π (lateral area + top + bottom) in two different ways: 1) substitution, and 2) Lagrange multipliers.

$A = 600\pi = 2\pi r^2 + 2\pi r h$

$\Rightarrow 300 = r^2 + rh$

$h = \frac{300 - r^2}{r}$

maximize $V = \pi r^2 h$

(a) $V(r) = \pi r^2 \left(\frac{300 - r^2}{r}\right)$

$= 300\pi r - \pi r^3$

$V'(r) = 300\pi - 3\pi r^2$

$= 3\pi(100 - r^2)$

$V'(r) = 0 \Rightarrow r = 10$

$\Rightarrow h = \frac{300 - r^2}{r} = 20$

$V = \pi r^2 h$

$= 2000\pi$

(b) $\nabla V = \lambda \nabla A$

$\begin{pmatrix} 2\pi r h \\ \pi r^2 \end{pmatrix} = \lambda \begin{pmatrix} 4\pi r + 2\pi h \\ 2\pi r \end{pmatrix}$

$\pi r^2 = \lambda 2\pi r \Rightarrow \lambda = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$

$2\pi r h = \frac{r}{2}(4\pi r + 2\pi h)$

$4\pi r h = 4\pi r^2 + 2\pi r h$

$2\pi r h = 4\pi r^2$

$h = 2r$

$300 = r^2 + rh$

$= r^2 + 2r^2$

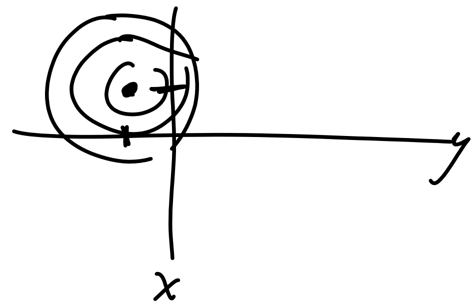
$= 3r^2$

$r = 10$

$h = 20$

3. Let $g(x, y) = -x^2 - y^2 - 2x - 2y - 2$. Find any critical points and identify as a local min/max or saddle point. Complete the square to identify the surface and verify your answer.

$$g(x,y) = -(x^2+2x+1) - (y^2+2y+1) - 2 + 2$$
$$= -(x+1)^2 - (y+1)^2$$



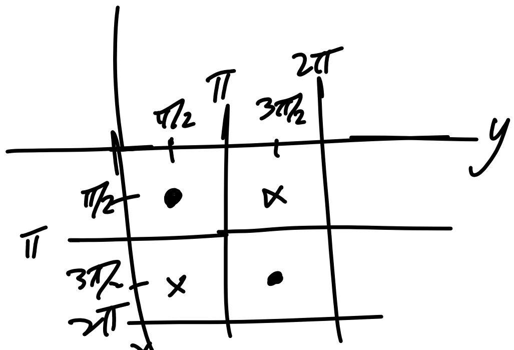
$g_x = -2x-2$ critical pts $x = -1 = y$

$g_y = -2y-2$ $d^2g = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ local max

$g_{xx} < 0$

4. Let $h(x, y) = \sin x \sin y$.

a) Verify that the following are all critical points: $(\pi/2, \pi/2), (\pi/2, 3\pi/2), (\pi, \pi), (3\pi/2, \pi/2), (3\pi/2, 3\pi/2)$



$h(x,y) = \sin x \sin y$

$h_x = \cos x \sin y$

$h_y = \sin x \cos y$

b) Classify each of the critical points using the 2nd derivative test.

$$d^2 h = \begin{pmatrix} -\sin x \sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{pmatrix}$$

$(\pi/2, \pi/2)$: $d^2 h = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ local max

$(3\pi/2, 3\pi/2)$

$(3\pi/2, \pi/2)$ $d^2 h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ local min

$\pi/2, 3\pi/2$

Bonus: Characterize the set of *all* critical points, and specify which are local minima, local maxima, and saddle points. Describe the shape of the graph $z = h(x, y)$.

Bonus: Write the 2nd order approximation for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

Bonus: Justify the 2nd derivative test for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

Bonus: Consider the line through (x_0, y_0) that is orthogonal to $\begin{pmatrix} a \\ b \end{pmatrix}$. Show that the point-normal form of the line is equivalent to the slope-point form of the line.

Bonus: Rotate the multiplication table to show that it is a pringle. In other words, rotate the function $m(x, y) = xy$ to remove the xy term and show that it is actually a hyperbolic paraboloid.