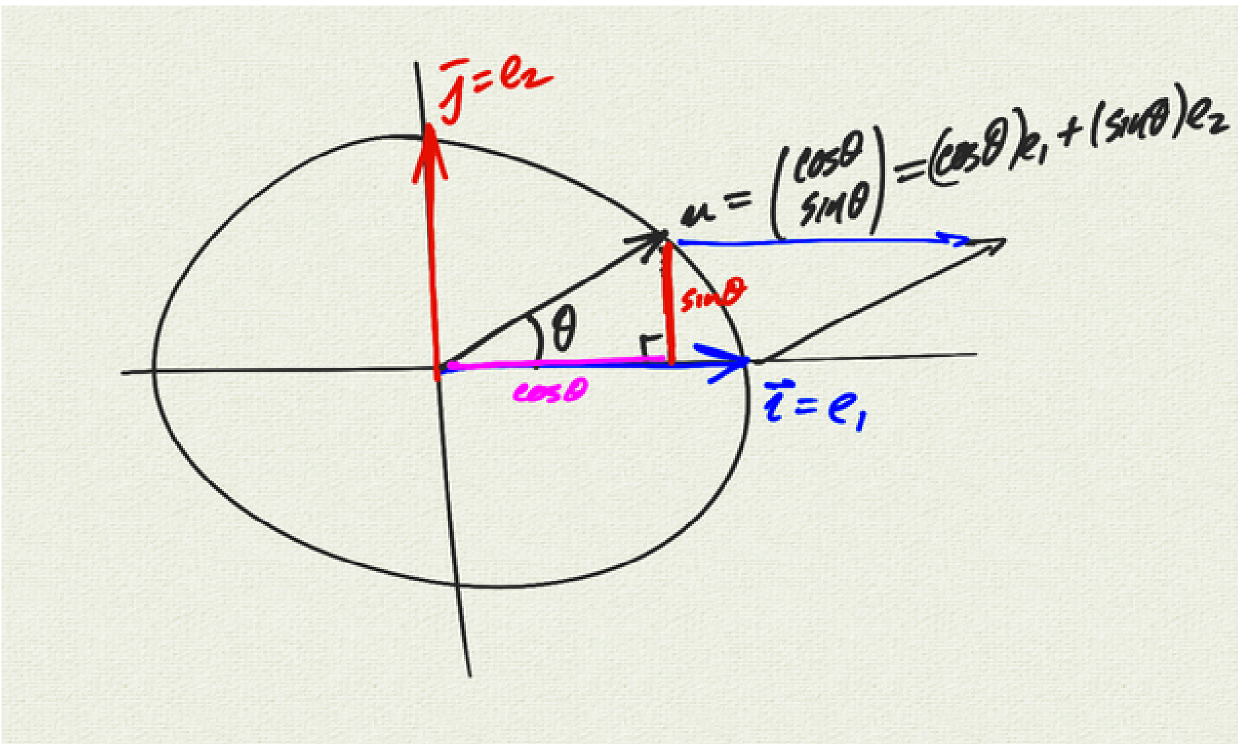
Geometric Algebra Notes 1 (Wedge Product)
MultiV 2021-22 / Dr. Kessner

The unit circle

First we're going to think about the unit circle in $\mathbb{R}^2$, and change our notation as well.

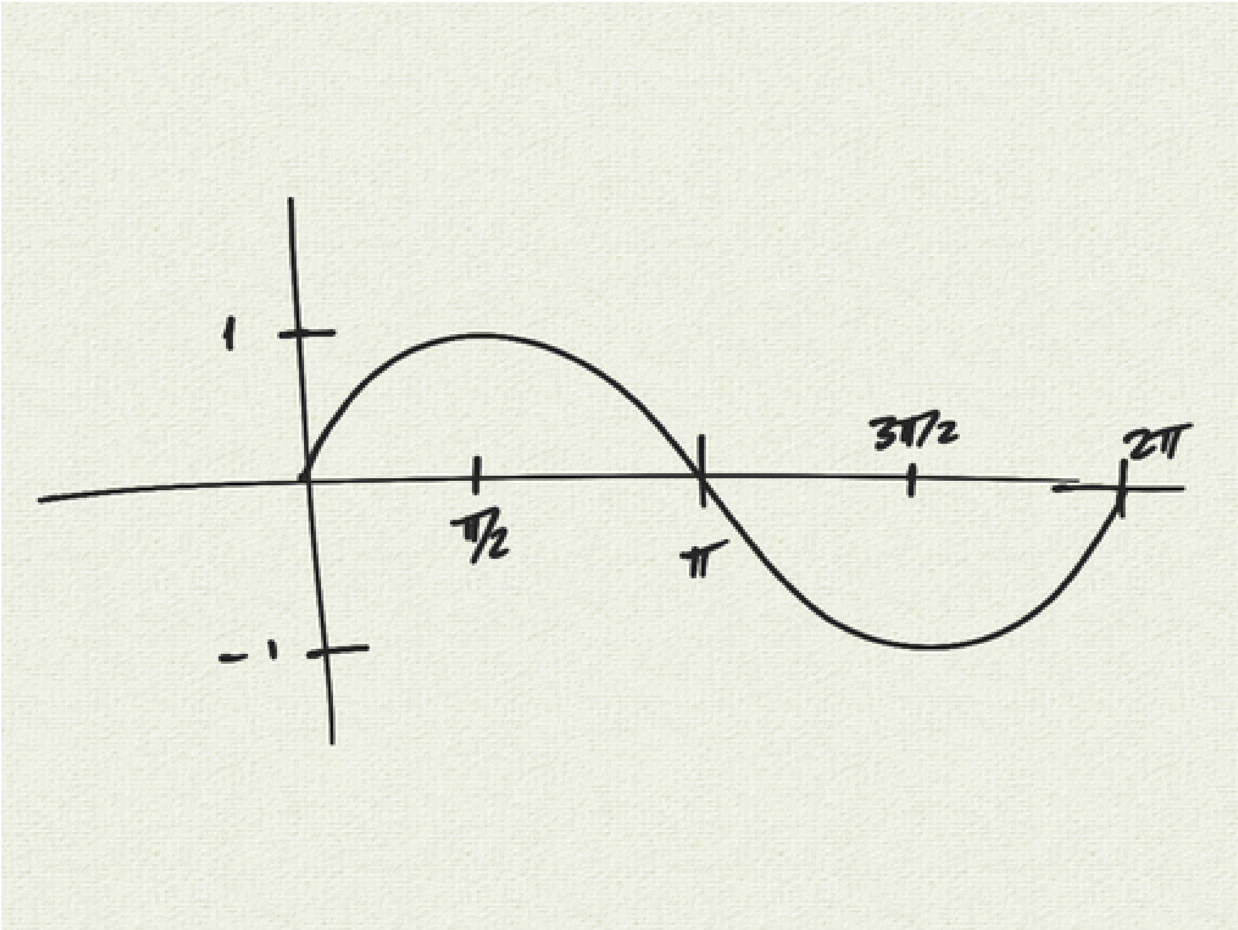
We're going to call our unit vectors $e_1 = \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let $u = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ be a vector on the unit circle.



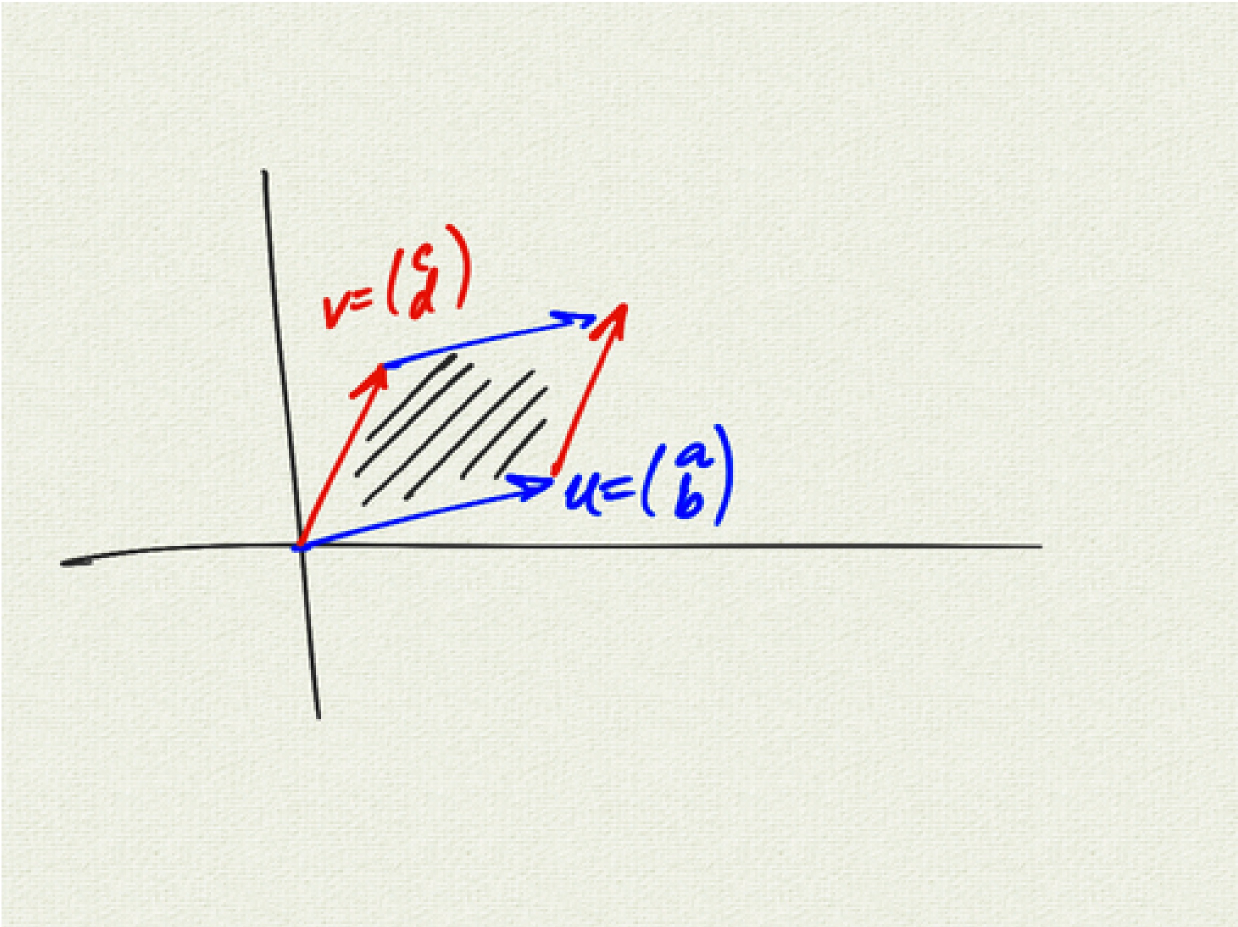
Observe that the projection of u on the x-axis is given by $\cos\theta$, and the area of the parallelogram determined by e_1 and u is $\sin\theta$.

We can graph the area of the parallelogram as u moves around the unit circle. Notice that for $\theta \in [\pi, 2\pi]$, the area is negative.


1
-1
π/2
π
3π/2
2π

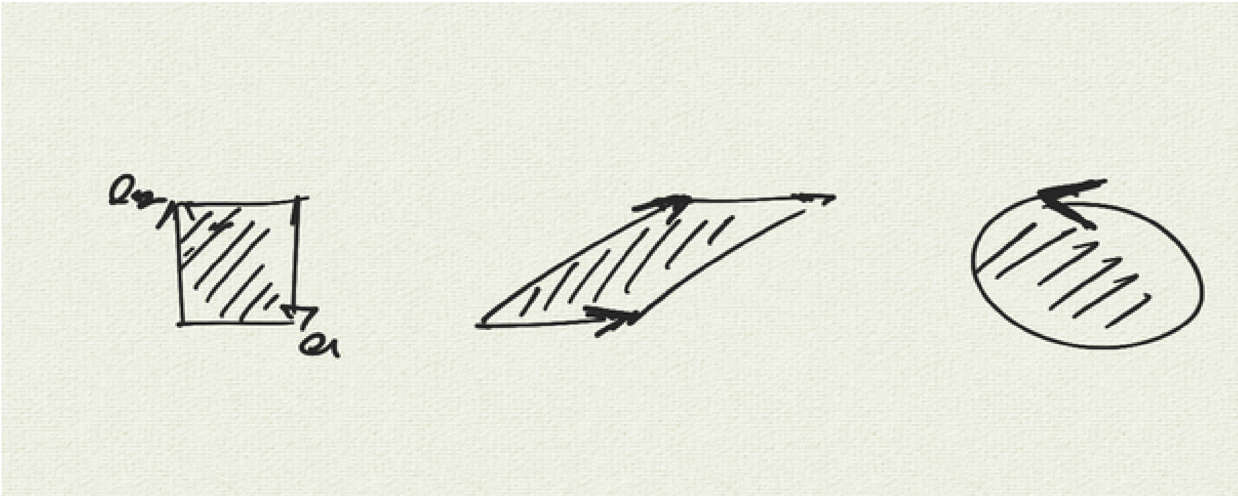
The wedge product

We have seen previously that $u = \begin{pmatrix} a \\ b \end{pmatrix}$ and $v = \begin{pmatrix} c \\ d \end{pmatrix}$, the area of the parallelogram is given by the determinant $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$.



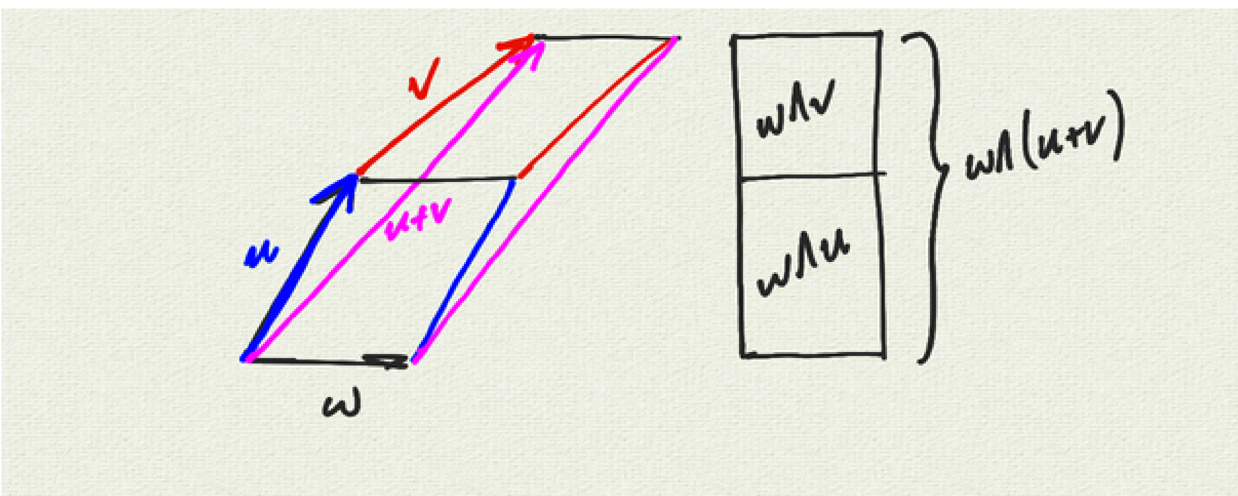
We define the wedge product $u \wedge v$ to be the directed (signed) area of the parallelogram determined by the two vectors, but with "units" (like meters^2). We call this directed area a *bivector*.

We define $e1 \wedge e2$ to be the "unit bivector". It represents the directed area of the square determined by e_1 and e_2. A general bivector will be a scalar multiple of $e_1 \wedge e_2$. However, the actual shape of of the bivector is not specified: we can think of it as a square, or reshape it to a parallelogram, or an amorphous shape in the plane.



From the definition of the wedge product, we observe that:
$e_1 \wedge e_1 = 0 = e_2 \wedge e_2$
$e_2 \wedge e_1 = -e_1 \wedge e_2$.

The distributive property is not so obvious:
$w \wedge (u + v) = w \wedge u + w \wedge v$



Once we believe the distributive property, we can do FOIL.
Let $u = \begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2$
and $v = \begin{pmatrix} c \\ d \end{pmatrix} = ce_1 + de_2$.

Then

$$\begin{aligned} u \wedge v &= (ae_1 + be_2) \wedge (ce_1 + de_2) \\ &= (ae_1 \wedge ce_1) + (ae_1 \wedge de_2) + (be_2 \wedge ce_1) + (be_2 \wedge de_2) \\ &= ac(e_1 \wedge e_1) + +ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1) + bd(e_2 \wedge e_2) \\ &= (ad - bc)(e_1 \wedge e_2) \end{aligned}$$

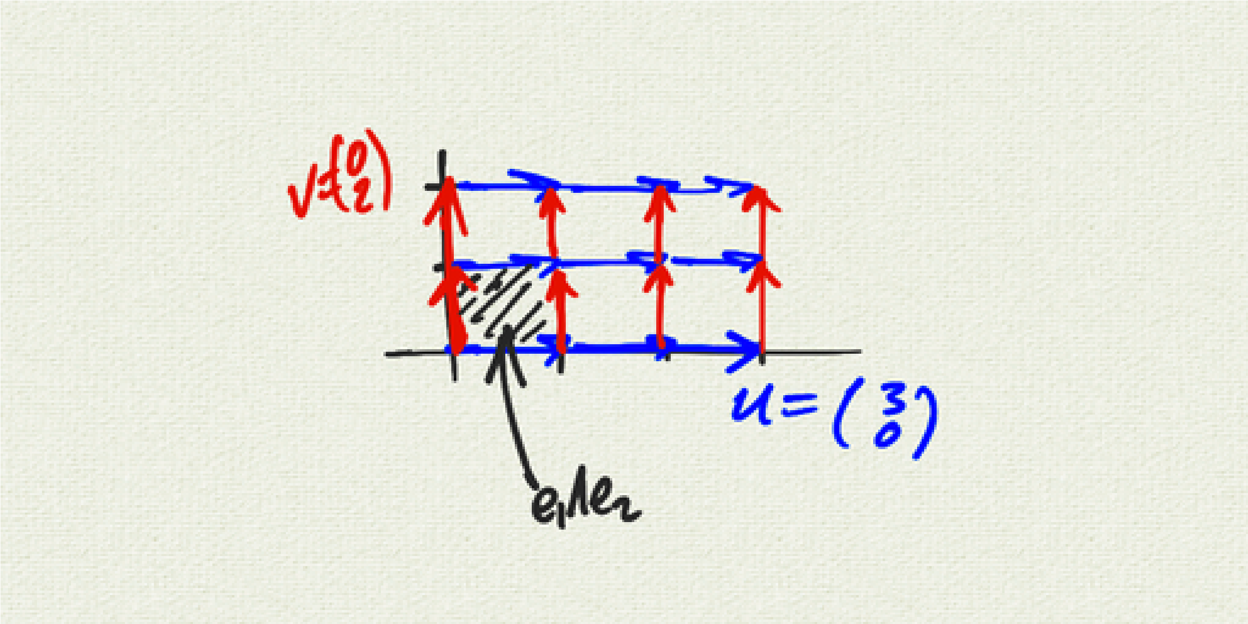
Notice that the determinant $ad - bc$ emerges as a consequence of the elementary properties of the wedge product.

Or we can use this as a shortcut for calculating the wedge product between two vectors:

$$
\begin{aligned}
u \wedge v &= \begin{pmatrix} a \\ b \end{pmatrix} \wedge \begin{pmatrix} c \\ d \end{pmatrix} \\
&= \begin{vmatrix} a & c \\ b & d \end{vmatrix} (e_1 \wedge e_2) \\
&= (ad - bc)(e_1 \wedge e_2)
\end{aligned}
$$

Here's an example:

$$
\begin{aligned}
\begin{pmatrix} 3 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 2 \end{pmatrix} &= \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} (e_1 \wedge e_2) \\
&= 6(e_1 \wedge e_2)
\end{aligned}
$$

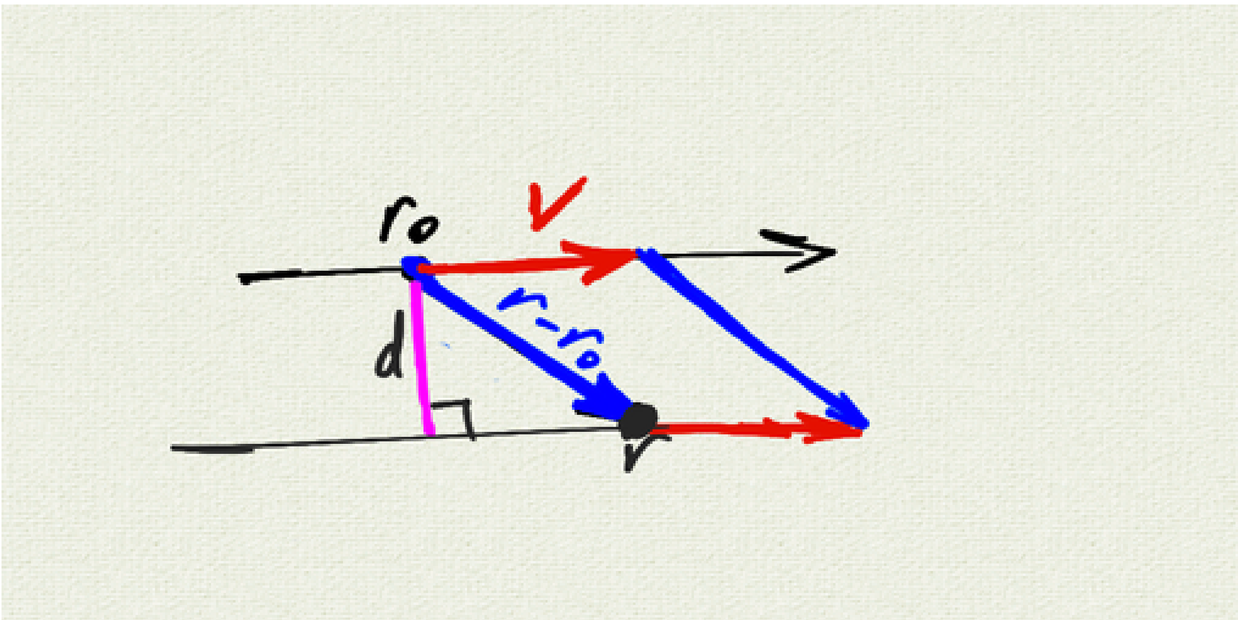


Application: Distance from a point to a line

Here's an application to a problem we have solved with the dot product and projection / rejection before: calculate the distance from a point to a line.

Suppose you have a line given by a point r_0 and a vector v. Suppose also that you have a point r in the plane. To calculuate the distance from the point to the line, you can find the area of the parallelogram between the vectors $r - r_0$ and v, and divide by the base (the length of v). The height of the parallelogram is the perpendicular distance from the point to the line:

$$d = \frac{|(r - r_0) \wedge v|}{|v|}$$



Note that you could do this equivalently with the cross product, but the geometry is not as obvious.