Unit 6 Group Work 2
PCHA 2022-23 / Dr. Kessner

KEY

No Calculator

1. Evaluate:

a. $\binom{7}{1} = 7$

b. $\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$

c. $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$

d. $\binom{7}{4} = \binom{7}{3} = 35$

e. $\binom{12}{2} = \frac{12 \cdot 11}{2} = 66$

f. $\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2} = 220$

g. $\binom{12}{9} = \binom{12}{3} = 220$

h. $\binom{12}{10} = \binom{12}{2} = 66$

i. $\binom{100}{99} = \binom{100}{1} = 100$

j. $\binom{2000}{2} = \frac{2000 \cdot 1999}{2} = 1999000$

2. Let $\{a_k\}_{k=1}^{\infty} = \{\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots\}$.

a. What type of sequence is this? Write recursive and explicit formulas for a_k.

geometric

$$a_{n+1} = \left(-\frac{1}{2}\right)a_n \quad \text{recursive}$$
$$a_1 = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} \quad \text{explicit}$$

b. Let S_n be the n^{th} partial sum of the sequence $\{a_k\}$. Express S_n (for this particular sequence) in summation notation.

$$S_n = \sum_{k=1}^{n} a_k$$
$$= \sum_{k=1}^{n} \frac{1}{2}\left(-\frac{1}{2}\right)^{k-1}$$

c. Write a formula for the actual sum S_n (for this particular sequence).

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$= \frac{\left(\frac{1}{2}\right)\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)}$$
$$= \frac{1}{3}\left(1-\left(-\frac{1}{2}\right)^n\right)$$

d. What is the sum of the infinite series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$? (Surprising?)

$$S_\infty \left(= \lim_{n\to\infty} S_n\right)$$
$$= \frac{a_1}{1-r}$$
$$= \frac{1}{3}$$

3. Expand $(2 - x^2)^4$.

$$= 2^4 + 4(2^3)(-x^2)^1 + 6(2^2)(-x^2)^2 + 4(2^1)(-x^2)^3 + (-x^2)^4$$

$$= 16 - 32x^2 + 24x^4 - 8x^6 + x^8$$

Find the x^6 term in $(2 - x^2)^5$.

$$\binom{5}{2} 2^2 (-x^2)^3$$

$$= -40x^6$$

Find the x^8 term in $(2 - x^2)^5$.

$$= \binom{5}{1} 2^1 (-x^2)^4$$

$$= 10x^8$$

4. Suppose you have 7 red and 3 white marbles in a bag. You pick 6 of the marbles from the bag (without replacement).

a. What is the probability that you pick 6 red marbles?

$$P(6\,\text{red}) = \frac{\binom{7}{6}\binom{3}{0}}{\binom{10}{6}} = \frac{7}{210} = \frac{1}{30}$$

$$\binom{10}{6} = \binom{10}{4} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 210$$

b. What is the probability that you pick 4 red (and 2 white marbles)?

$$P(4\,\text{red}) = \frac{\binom{7}{4}\binom{3}{2}}{\binom{10}{6}} = \frac{35 \cdot 3}{210} = \frac{105}{210} = \frac{1}{2}$$

$$\binom{7}{4} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$$

c. What is the probability that you pick 2 red marbles?

$$P(2\,\text{red}) = \binom{7}{2}\binom{3}{4} = 0$$

5. Suppose you have 9 black and 3 white marbles in a bag. replacement $\rightarrow$ binomial

a. You sample 5 marbles with replacement (in other words, you pick a marble, look at it, and put it back, 5 times). Let B be the number of times you pick a black marble. Calculate all of the 6 probabilities $P(B=0), P(B=1), ..., P(B=5)$. Verify that $1 = \sum_{k=0}^{5} P(B=k)$.

$P(B=0) = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

$P(B=1) = 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 = \frac{15}{1024}$

$P(B=2) = 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 = \frac{90}{1024}$

$P(B=3) = 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = \frac{270}{1024}$

$P(B=4) = 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 = \frac{405}{1024}$

$P(B=5) = \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$

1
15
90
270
405
243
1024 ✓

no replacement $\rightarrow$ hypergeometric

b. This time you sample 5 marbles without replacement (in other words, you grab 5 marbles from the bag, all at the same time). Calculate the probabilities $P(B=0), P(B=1), ..., P(B=5)$ and again verify that $1 = \sum_{k=0}^{5} P(B=k)$.

9 Black
3 white
grab 5

$P(0) = 0$

$P(1) = 0$

$P(2) = \frac{\binom{9}{2}\binom{3}{3}}{\binom{12}{5}} = \frac{36}{792}$

$P(3) = \frac{\binom{9}{3}\binom{3}{2}}{\binom{12}{5}} = \frac{84 \cdot 3}{} = \frac{252}{792}$

$P(4) = \frac{\binom{9}{4}\binom{3}{1}}{\binom{12}{5}} = \frac{126 \cdot 3}{} = \frac{378}{792}$

$P(5) = \frac{\binom{9}{5}\binom{3}{0}}{\binom{12}{5}} = \frac{126}{792}$

$\binom{12}{5} = \frac{12 \cdot 11 \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 132 \cdot 6 = 792$

$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{\cancel{3 \cdot 2}} = 84$

$\binom{9}{4} = \frac{9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6}}{\cancel{4 \cdot 3 \cdot 2}} = 126$

$\binom{9}{5} = \frac{9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot \cancel{5}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2}} = 126$

36
252
378
126
792 ✓