Unit 9 Group Work B
PCHA 2022-23 / Dr. Kessner

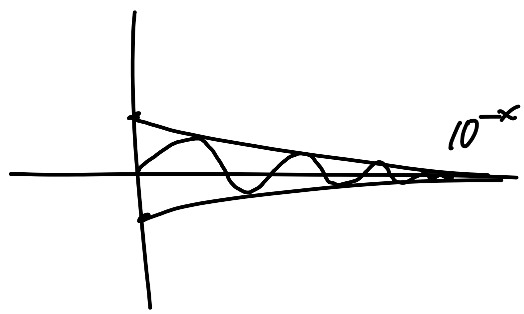
No calculator! Have fun!

1. Evaluate the following limits, evaluating left and right side limits where applicable.

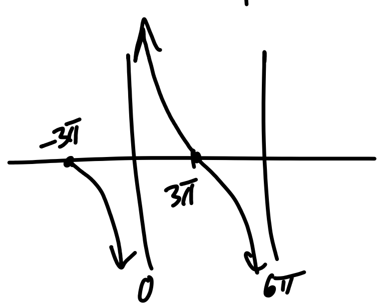
a. $\lim\limits_{x\to 0} x \csc \dfrac{x}{3}$ $= \lim\limits_{x\to 0} \underbrace{\dfrac{x}{\sin\left(\frac{x}{3}\right)} \cdot \dfrac{1/3}{}}_{\to 1}\!\!\dfrac{}{1/3} = 3$

b. $\lim\limits_{x\to 0} x \sin \dfrac{x}{3}$ $=0$

c. $\lim\limits_{x\to \infty} 10^{-x} \sin \dfrac{x}{3}$ $=0$



d. $\lim\limits_{x\to 0} \cot \dfrac{x}{3}$



$\lim\limits_{x\to 0^-} \cot \dfrac{x}{3} = -\infty$

$\lim\limits_{x\to 0^+} \cot \dfrac{x}{3} = +\infty$

e. $\lim\limits_{x\to 0} \dfrac{\sin(\pi + x) - \sin(\pi)}{x}$. $= f'(\pi)$ with $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$f'(\pi) = \cos\pi = -1$

2. a. Find the derivative of $f(x) = \cos 2x$ using a limit definition. Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.
Hint: Use the sum angle formula $\cos(u+v) = \cos u \cos v - \sin u \sin v$, but don't use the double angle formula.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cos[2(x+h)] - \cos 2x}{h}$$
$$= \lim_{h\to 0} [\cos(2x+2h) - \cos 2x]\frac{1}{h}$$
$$= \lim_{h\to 0} [\cos 2x \cos 2h - \sin 2x \sin 2h - \cos 2x]\frac{1}{h}$$
$$= \lim_{h\to 0} \Big[\cos 2x \underbrace{\left(\frac{\cos 2h - 1}{h}\right)}_{\to 0} - \sin 2x \underbrace{\frac{\sin 2h}{h}\cdot\frac{2}{2}}_{\to 1}\Big]$$
$$= -2\sin 2x$$

b. Find the derivative of $g(x) = \frac{1}{x}$ using the limit definition:

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a}$$

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a}$$
$$= \lim_{x\to a} \frac{1/x - 1/a}{x - a}$$
$$= \lim_{x\to a} \left(\frac{a - x}{ax}\right)\frac{1}{x - a}$$
$$= \lim_{x\to a} -\frac{1}{ax}$$
$$= -\frac{1}{a^2}$$

$g'(a) = -\frac{1}{a^2}$ (for all a) $\Longrightarrow$ $g'(x) = -\frac{1}{x^2}$ (for all x)

3. Using the various rules for differentiation, calculate the derivatives of the following functions.

a. $p(x) = e^{\sin x}$

$$p'(x) = e^{\sin x} \cos x$$

b. $q(x) = \sin^2 x + \cos^2 x$ (Practice using power and chain rules!)

$$q'(x) = 2\sin x \cos x + 2\cos x(-\sin x) = 0$$

c. $r(x) = \sin^4 x - \cos^4 x$

$$\begin{aligned} r'(x) &= 4\sin^3 x \cos x - 4\cos^3 x(-\sin x) \\ &= 4\sin x \cos x(\sin^2 x + \cos^2 x) \\ &= 4\sin x \cos x \end{aligned}$$

d. $s(x) = -\cos 2x$ (Notice that $s'(x) = r'(x)$. Challenge: verify that $r(x) = s(x)$.)

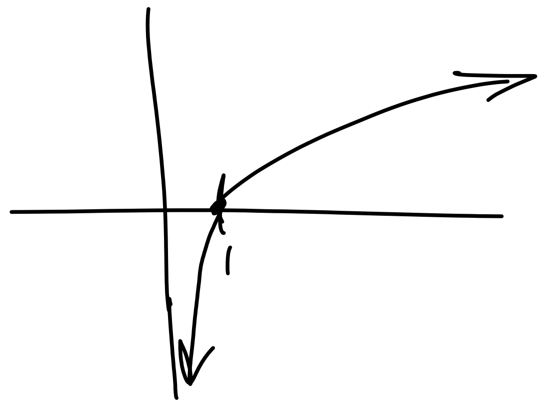
$$s'(x) = +2\sin 2x \qquad (= 4\sin x \cos x)$$

e. $t(x) = 2^{\sin x^2}$

$$t'(x) = 2^{\sin x^2} \ln 2 \cdot \cos x^2 (2x)$$

4. Consider the curve $x = 10^y$.

a. Sketch the graph of this curve.



$x = 10^y$

$y = \log_{10} x$

b. Find $\frac{dy}{dx}$ (in terms of x and y) by implicit differentiation.

$x = 10^y$

$1 = 10^y \ln 10 \, dy/dx$

$\frac{dy}{dx} = \frac{1}{10^y \ln 10}$

c. Solve for y in terms of x.

$y = \log_{10} x$

d. Find $\frac{dy}{dx}$ using the expression for y you found above.

$\frac{dy}{dx} = \frac{1}{x \ln 10}$

e. Verify that these two formulas for $\frac{dy}{dx}$ are the same. $x = 10^y \Longrightarrow$

$\frac{dy}{dx} = \frac{1}{x \ln 10} = \frac{1}{10^y \ln 10}$ ✓

5. Suppose you have 128 kg of ^{14}C, which has a half-life of 5730 years.

a. Write an equation to model the amount $A(t)$ of ^{14}C as a function of time.

t	$A(t)$
0	128
5730	64
$2 \cdot 5730$	32
$3 \cdot 5730$	16

$$A(t) = 128\left(\frac{1}{2}\right)^{t/5730}$$

b. Find the average rate of change in the amount over the first 5 half-lives ($5 \cdot 5730$ years). Use a calculator to get approximate values.

$$\frac{A(5 \cdot 5730) - A(0)}{5 \cdot 5730} = \frac{128\left(\frac{1}{2}\right)^5 - 128}{5}$$

$$= \frac{4 - 128}{5 \cdot 5730} \approx -.004$$

c. Find $A'(t)$.

$$A(t) = 128\left(\frac{1}{2}\right)^{t/5730}$$

$$A'(t) = 128\left(\frac{1}{2}\right)^{t/5730} \ln\frac{1}{2} \cdot \frac{1}{5730}$$

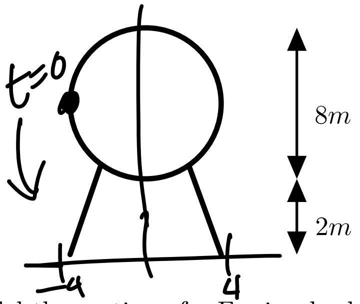
d. Calculate the rate of change (exact) at $t = 0$, $t = 2 \cdot 5730$, and $t = 5 \cdot 5730$ years. Use a calculator to get approximate values.

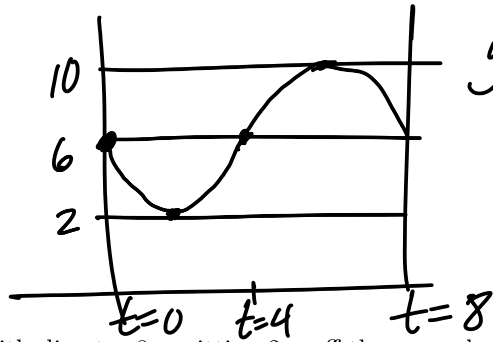
$$A'(0) = \frac{128}{5730}\left(\ln\frac{1}{2}\right) \approx -.015$$

$$A'(2 \cdot 5730) = \frac{128}{5730}\left(\ln\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 \approx -.0039$$

$$A'(5 \cdot 5750) = \frac{128}{5730}\left(\ln\frac{1}{2}\right)\left(\frac{1}{2}\right)^5 \approx -.0005$$

6.

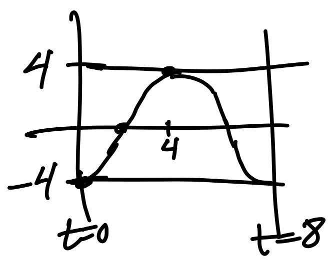




$y(t) = 6 - 4\sin\left(\frac{2\pi}{8}t\right)$

Model the motion of a Ferris wheel with diamter 8m, sitting 2m off the ground. Suppose you start ($t = 0$) at the 9 o'clock position (furthest left on diagram), traveling counter-clockwise, and that the period is 8 minutes.

a. Write parametric equations $x(t)$ and $y(t)$ to model the position as a function of time.



$x(t) = -4\cos\left(\frac{2\pi}{8}t\right)$

$x(t) = -4\cos\left(\frac{\pi}{4}t\right)$

$y(t) = 6 - 4\sin\left(\frac{\pi}{4}t\right)$

b. Find $x'(t)$ and $y'(t)$.

$$x'(t) = +4\sin\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4}$$
$$= \pi\sin\left(\frac{\pi}{4}t\right)$$
$$y'(t) = -\pi\cos\left(\frac{\pi}{4}t\right)$$

c. Evaluate $x'(t)$ and $y'(t)$ at the bottom position. $t = 2$

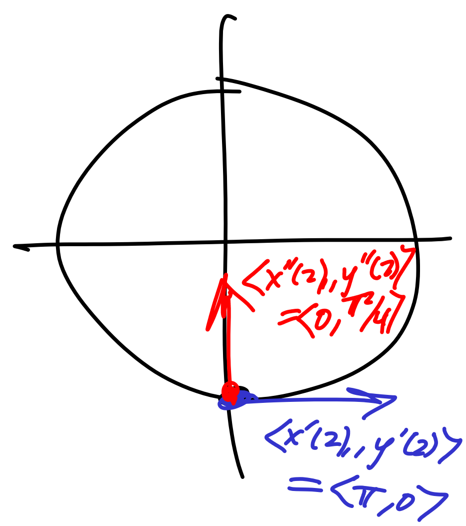
$$x'(2) = \pi\sin\frac{\pi}{2} = \pi$$
$$y'(2) = -\pi\cos\frac{\pi}{2} = 0$$

d. Find $x''(t)$ and $y''(t)$.

$$x''(t) = \frac{\pi^2}{4}\cos\left(\frac{\pi}{4}t\right)$$
$$y''(t) = +\frac{\pi^2}{4}\sin\left(\frac{\pi}{4}t\right)$$

e. Evaluate $x''(t)$ and $y''(t)$ at the bottom position.

$$x''(2) = \frac{\pi^2}{4}\cos\frac{\pi}{2} = 0$$
$$y''(2) = \frac{\pi^2}{4}\sin\frac{\pi}{2} = \frac{\pi^2}{4}$$

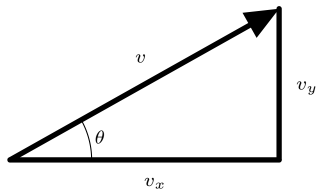


7. Recall that you can model projectile motion with parametric equations:

$$x(t) = x_0 + v_x t$$

$$y(t) = y_0 + v_y t - 16t^2$$

where (x_0, y_0) is the initial position of the object, and v_x and v_y are the components of the initial velocity vector v:



Suppose that you launch a rocket straight up from the ground (i.e. at an angle of $\frac{\pi}{2}$), with an initial speed of 96 ft/sec.

$V_x = v\cos\frac{\pi}{2} = 0$
$V_y = v\sin\frac{\pi}{2} = 96$

a. Write equations for $x(t)$ and $y(t)$.

$x(t) = 0$
$y(t) = 96t - 16t^2$

b. Find $x'(t)$ and $y'(t)$.

$x'(t) = 0$
$y'(t) = 96 - 32t$

c. Find $x''(t)$ and $y''(t)$.

$x''(t) = 0$
$y''(t) = -32$

d. Using the derivatives you found above, find the maximum height of the rocket.

$y'(t) = 0 \Rightarrow 96 - 32t = 0$
$t = 3$

$y(3) = 96(3) - 16 \cdot 3^2 = 144$

e. When does the rocket hit the ground, and how far has it traveled in the x-direction?

$y(t) = 0 \Rightarrow$

$96t - 16t^2 = 0$
$-16t(t-6) = 0$
$t = 0, 6$
6 secs.
$x(6) = 0$