KEY

Unit 9 Group Work
PCHA 2022-23 / Dr. Kessner

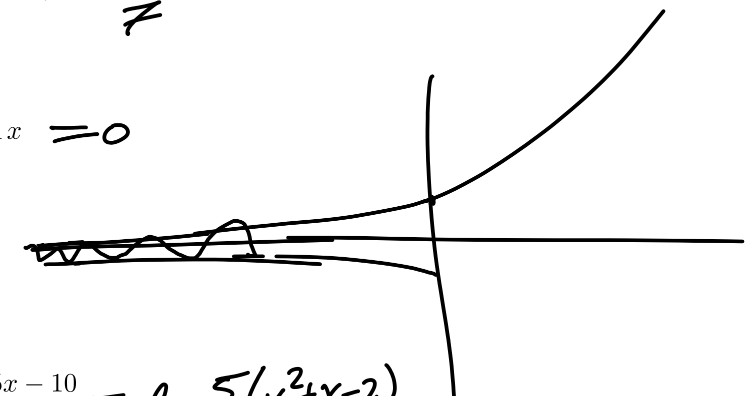
No calculator! Have fun!

1. Evaluate the following limits, evaluating left and right side limits where applicable.

a. $\lim_{x\to 0} x\cot 7x$ $= \lim_{x\to 0} x \cdot \frac{\cos 7x}{\sin 7x} \cdot \frac{7}{7}$ $\to 1$

$= \frac{1}{7}$

b. $\lim_{x\to -\infty} e^x \sin x$ $= 0$

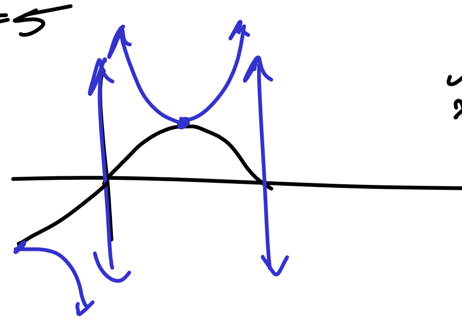


c. $\lim_{x\to 1} \dfrac{5x^2+5x-10}{(x-1)(x+2)}$ $= \lim_{x\to 1} \dfrac{5(x^2+x-2)}{(x-1)(x+2)}$

$= 5 \lim_{x\to 1} \dfrac{(x+2)(x-1)}{(x+2)(x-1)}$

$= 5$

d. $\lim_{x\to 0} \csc x$ $= \lim_{x\to 0} \frac{1}{\sin x}$



$\lim_{x\to 0^+} \csc x = +\infty$

$\lim_{x\to 0^-} \csc x = -\infty$

e. $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$, where $f(x)=x^2$. *(Hint: use what you know about derivatives)*

$= f'(3)$

$f'(x) = 2x$

$f'(3) = 6$

2. For the following functions find the derivative using one of the limit definitions.

a. Suppose that a little bird or a mathematician tells you that $\lim_{h\to 0} \dfrac{a^h - 1}{h} = \ln(a)$. Find the derivative of $f(x) = a^x$ (using a limit definition).

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= a^x \ln a$$

b. Find $g'(x)$, where $g(x) = mx + b$, using a limit definition.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{m(x+h) + b - (mx+b)}{h}$$

$$= \lim_{h\to 0} \frac{mh}{h}$$

$$= m$$

3. Using the various rules for differentiation, calculate the derivatives of the following functions.

a. $p(x) = \tan x \cot x$ $= 1$

$\Rightarrow p'(x) = 0$

or: $p'(x) = \sec^2 x \cot x + \tan x(-\csc^2 x)$

$= \frac{1}{\cos^2 x}\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\frac{1}{\sin^2 x}$

$= 0$

b. $q(x) = 2 \sin x \cos x.$ $= \sin 2x$

$q'(x) = 2\cos 2x$

or: $q'(x) = 2(\cos x \cos x + \sin x(-\sin x))$

$= 2(\cos^2 x - \sin^2 x)$

c. $r(x) = \sin 2x + \sin^2 x + \sin x^2$

$r'(x) = 2\cos 2x + 2\sin x \cos x + 2x \cos x^2$

d. $s(x) = e^{\cot(x^3-1)}$

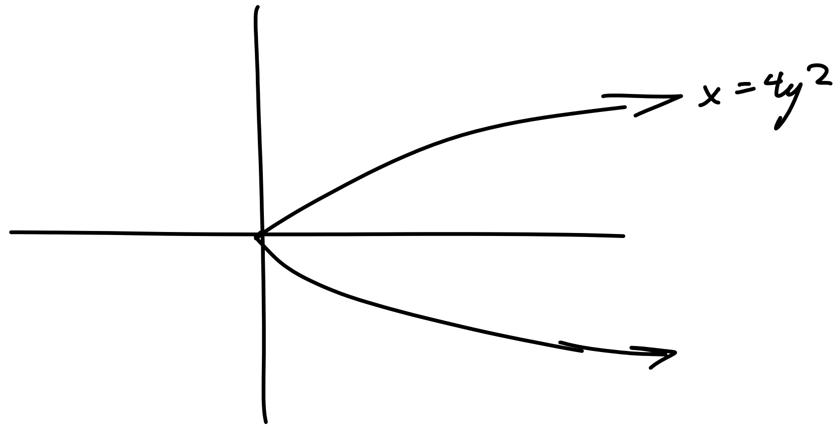
$s'(x) = e^{\cot(x^3-1)} \cdot (-\csc^2(x^3-1)) \cdot (3x^2)$

e. $t(x) = \log_2(\sec^3(x^5))$

$t'(x) = \frac{1}{\sec^3(x^5)\ln 2} \cdot 3\sec^2(x^5) \cdot \sec(x^5)\tan(x^5) \cdot 5x^4$

4. Consider the curve $x = 4y^2$.

a. Sketch the graph of this curve.



b. Find $\frac{dy}{dx}$ (in terms of x and y) by implicit differentiation.

$$x = 4y^2$$
$$1 = 8y\frac{dy}{dx} \longrightarrow \frac{dy}{dx} = \frac{1}{8y}$$

c. Solve for y in terms of x (choose the positive square root).

$$x = 4y^2 \longrightarrow y = \sqrt{x/4} = \frac{1}{2}x^{1/2}$$

d. Find $\frac{dy}{dx}$ using the expression for y you found above.

$$\frac{dy}{dx} = \frac{1}{4}x^{-1/2} = \frac{1}{4\sqrt{x}}$$

e. Verify that these two formulas for $\frac{dy}{dx}$ are the same.

$$\frac{1}{8y} = \frac{1}{8\cdot\frac{1}{2}\sqrt{x}} = \frac{1}{4\sqrt{x}} \quad ✓$$

5. Suppose a bacterial colony begins with 4000 cells and the population doubles every 4 hours.

a. Write an equation to model the population $P(t)$ of the colony as a function of time.

$$P(t) = 4000 \cdot 2^{t/4}$$

t	$P(t)$
0	4000
4	8000
8	16000

b. Find the average rate of growth in the population over the first 8 hours.

$$\frac{P(8)-P(0)}{8} = \frac{12000}{8} = 1500$$

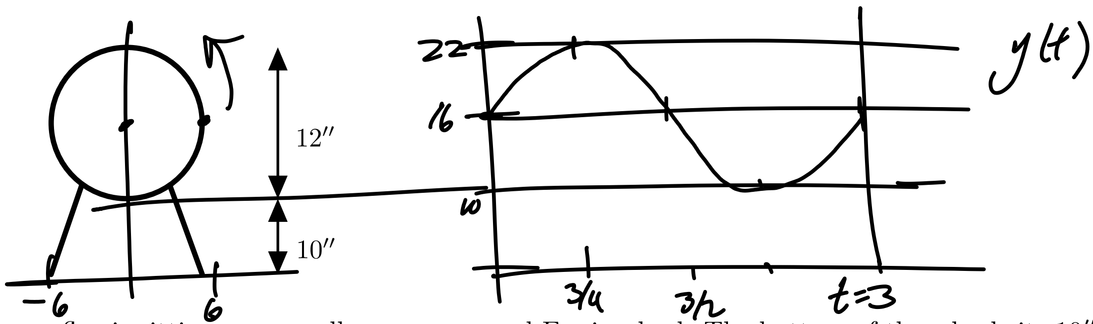
c. Find $P'(t)$.

$$P'(t) = 4000 \cdot 2^{t/4} \cdot \ln 2 \left(\tfrac{1}{4}\right)$$
$$= (1000 \ln 2) 2^{t/4}$$

d. Calculate the growth rate (exact) at $t = 0$, $t = 4$, and $t = 8$ hours. Given that $\ln 2 \approx .693$, approximate these rates (calculator ok).

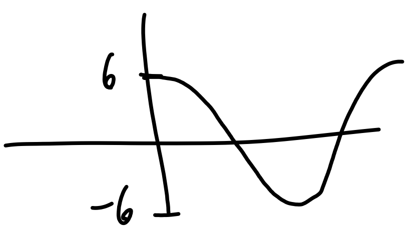
$$P'(0) = 1000 \ln 2 \approx 693 \approx 700$$
$$P'(4) = 2000 \ln 2 \approx 1400$$
$$P'(8) = 4000 \ln 2 \approx 2800$$

6.



Suppose a flea is sitting on a small mouse-powered Ferris wheel. The bottom of the wheel sits 10″ off the ground, and the diameter of the wheel is 12″. You give the mouse some coffee so the mouse runs fast: 3 seconds for a revolution. The flea starts at the point furthest to the right, and the wheel moves counter-clockwise.

a. Write parametric equations $x(t)$ and $y(t)$ to model the position of the flea as a function of time. When will the flea first be at the top of the wheel? Verify the position of the flea at that time.



$x(t) = 6\cos\left(\frac{2\pi}{3}t\right)$

$y(t) = 16 + 6\sin\left(\frac{2\pi}{3}t\right)$

top: $t = 3/4$

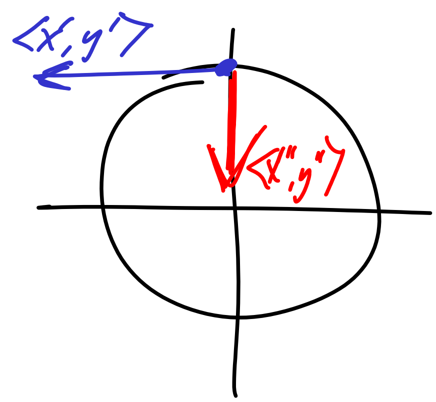
$x(3/4) = 0$

$y(3/4) = 22$

b. Find $x'(t)$ and $y'(t)$.

$x'(t) = -6\sin\left(\frac{2\pi}{3}t\right) \cdot \frac{2\pi}{3} = -4\pi\sin\left(\frac{2\pi}{3}t\right)$

$y'(t) = 4\pi\cos\left(\frac{2\pi}{3}t\right)$



c. Evaluate $x'(t)$ and $y'(t)$ at the top of the wheel.

$x'\left(\frac{3}{4}\right) = -4\pi$

$y'\left(\frac{3}{4}\right) = 0$

d. Find $x''(t)$ and $y''(t)$.

$x''(t) = -\frac{8\pi^2}{3}\cos\left(\frac{2\pi}{3}t\right)$

$y''(t) = -\frac{8\pi^2}{3}\sin\left(\frac{2\pi}{3}t\right)$

e. Evaluate $x''(t)$ and $y''(t)$ at the top of the wheel.

$x''\left(\frac{3}{4}\right) = 0$

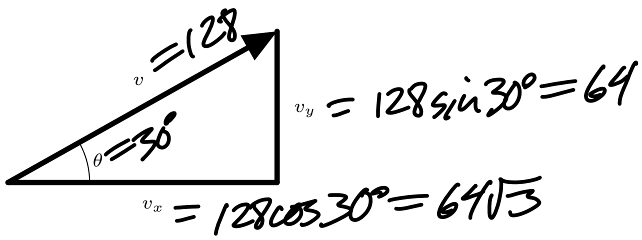
$y''\left(\frac{3}{4}\right) = -\frac{8\pi^2}{3}$

7. Recall that you can model projectile motion with parametric equations:

$$x(t) = x_0 + v_x t$$

$$y(t) = y_0 + v_y t - 16t^2$$

where (x_0, y_0) is the initial position of the object, and v_x and v_y are the components of the initial velocity vector v:



Suppose that you launch a rocket from an 80' building, at an angle of 30°, with an initial speed of 128 ft/sec.

a. Write equations for $x(t)$ and $y(t)$.

$x(t) = 64\sqrt{3}t$

$y(t) = 80 + 64t - 16t^2$

b. Find $x'(t)$ and $y'(t)$. Interpret your answer.

$x'(t) = 64\sqrt{3}$

$y'(t) = 64 - 32t$

c. Find $x''(t)$ and $y''(t)$. Interpret your answer.

$x''(t) = 0$

$y''(t) = -32$

d. Using the derivatives you found above, find the maximum height of the rocket.

$y'(t) = 0 \Rightarrow 64 - 32t = 0 \Rightarrow y(2) = 80 + 128 - 64 = 144$

$t = 2$

e. When does the rocket hit the ground, and how far has it traveled in the x-direction?

$y(t) = -16t^2 + 64t + 80$

$= -16(t^2 - 4t - 5)$

$= -16(t-5)(t+1)$

$y(t) = 0 \Rightarrow t = 5$

$x(5) = 64\sqrt{3} \cdot 5 = 320\sqrt{3}$