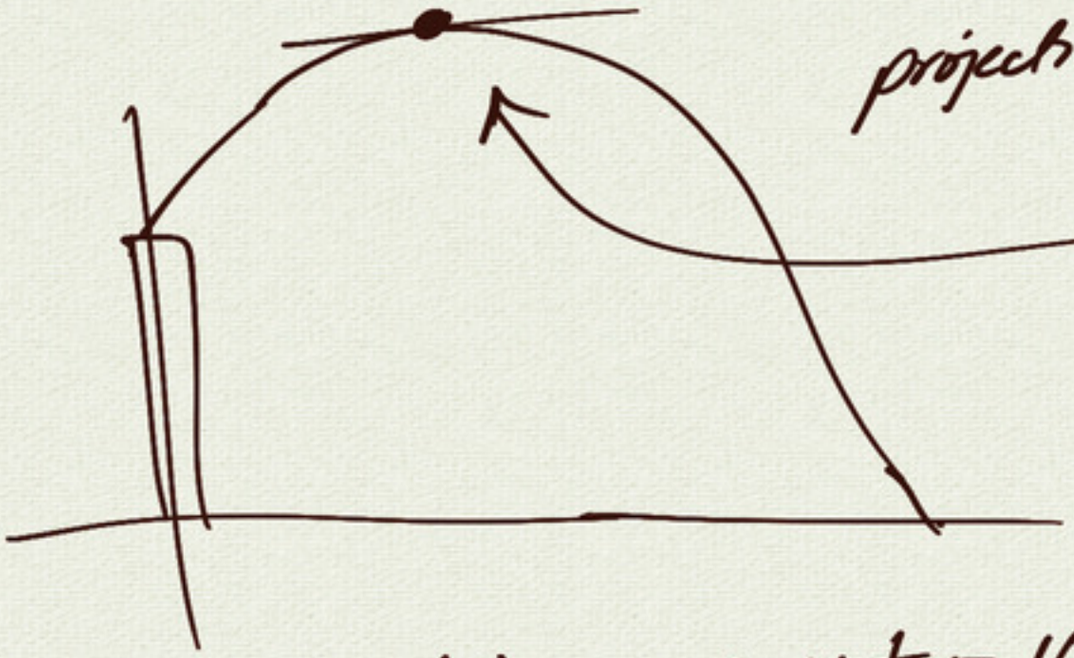
10.4 Optimization

find min or max



find max height
$y(t)$

$\Rightarrow$ look for $y'(t)=0$
critical pts.

$$y(t) = y_0 + v_y t - 16t^2$$

$$\Rightarrow y'(t) = v_y - 32t$$

(critical pt
$y'(t)=0$
$v_y = 32t$
$t_{max} = \frac{v_y}{32}$)

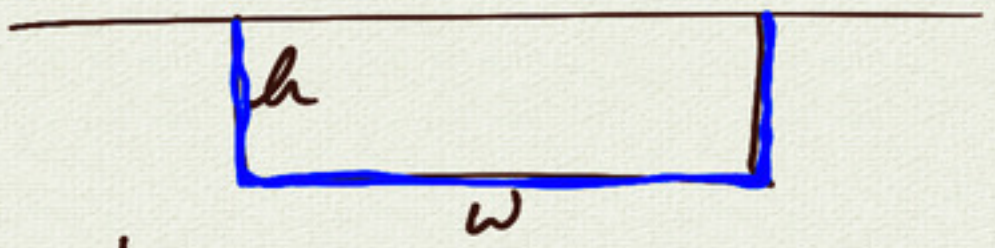
$$y''(t) = -32$$

$y''(t_{max}) = -32 < 0$ local max

example:

build fence with 100 m fencing material

maximize area (assume rectangle)



$A = w \cdot h$

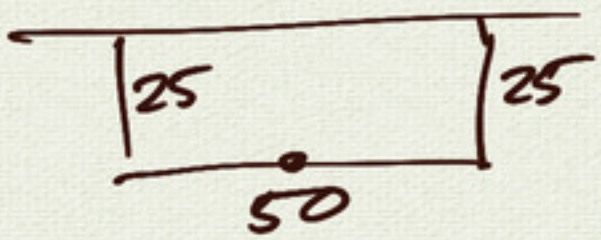
constraint: $2h + w = 100 \Rightarrow w = 100 - 2h$

$A = wh$

$\boxed{A(h) = (100-2h)h}$ maximize

$A = 100h - 2h^2$

$\Rightarrow \boxed{A'(h) = 100 - 4h}$



critical pts: $A'(h) = 0$

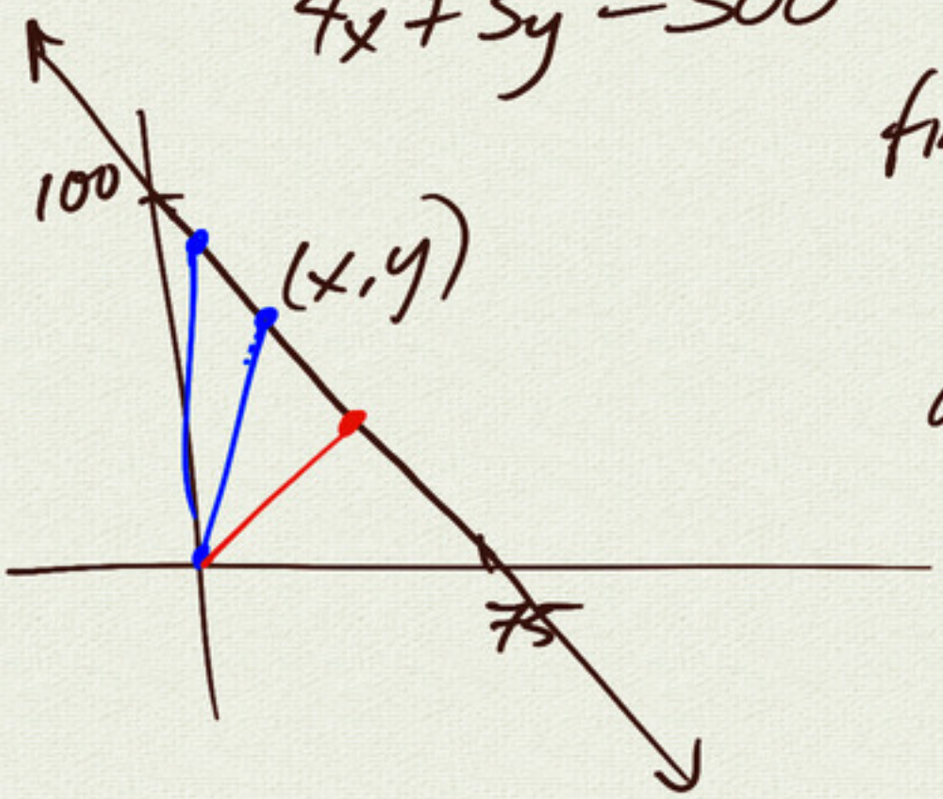
$100 - 4h = 0$

$h = 25 \Rightarrow w = 100 - 2(25) = 50$

$\boxed{A''(h) = -4}$

$A''(25) = -4 < 0$ ∩ local max

example: line $4x+3y=300$



point

find (x,y) on line closest to origin

$d^2 = x^2 + y^2$

$f(x) = x^2 + y^2$

minimize distance squared

$d_1 < d_2 \Leftrightarrow d_1^2 < d_2^2$

$4x+3y=300$

$y = \frac{1}{3}(300-4x)$

$\boxed{f(x) = x^2 + \frac{1}{3^2}(300-4x)^2}$ minimize

$f'(x) = 2x + \frac{1}{9}2(300-4x)(-4)$

$= 2x - \frac{8}{9}\cdot 300 + \frac{32}{9}x$

$= 2x - \frac{800}{3} + \frac{32}{9}x$

$f'(x) = 0 \Rightarrow \left(2+\frac{32}{9}\right)x = \frac{800}{3}$

$\frac{50}{9}x = \frac{800}{3}$

$x = 48$

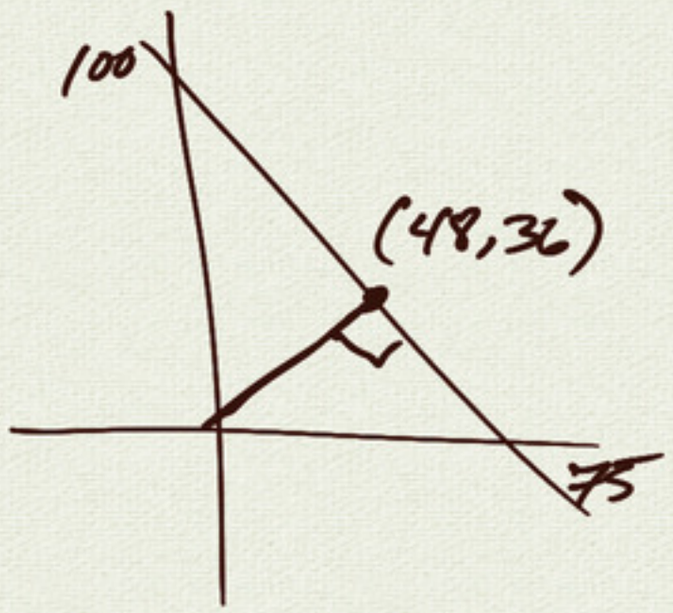
$y = \frac{1}{3}(300-4x)$

$= \frac{1}{3}(300-4\cdot 48)$

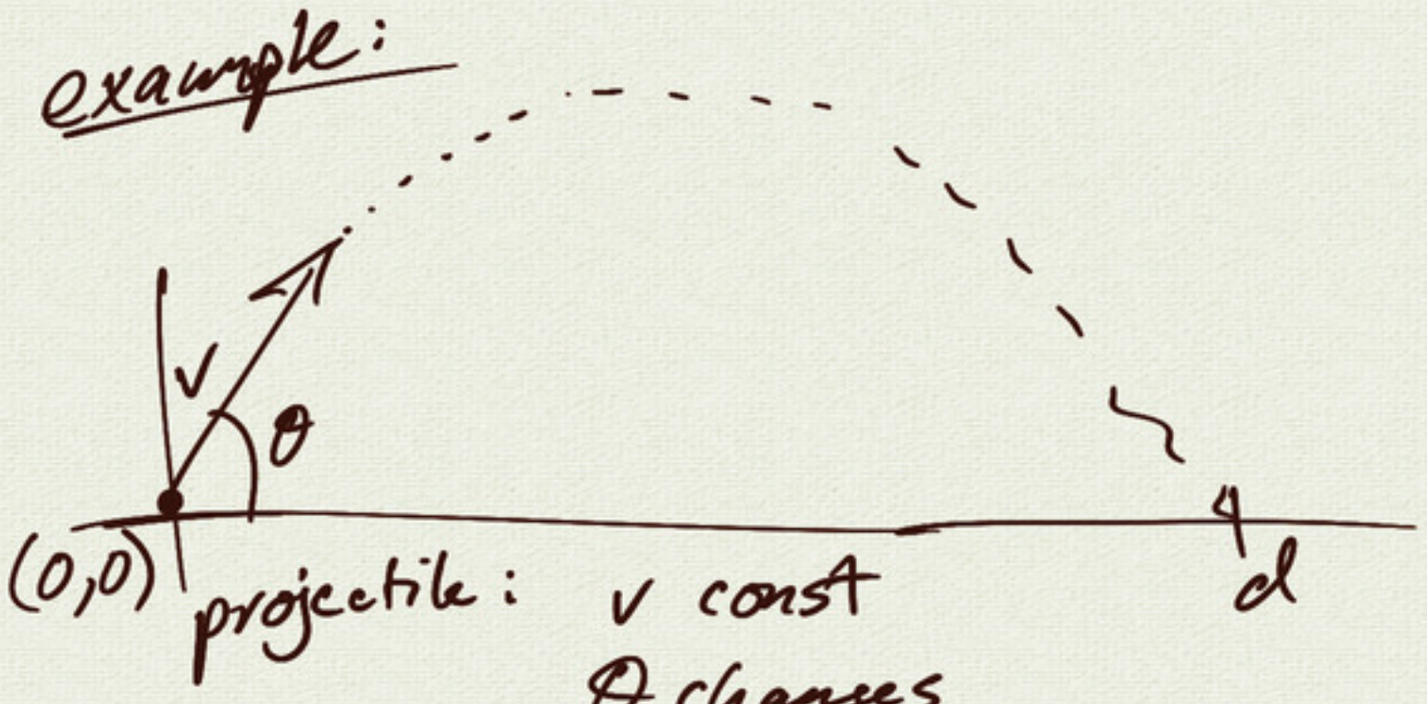
$= 36$

$f''(x) = 2 + \frac{32}{9}$

$f''(48) > 0$ ∪ local min



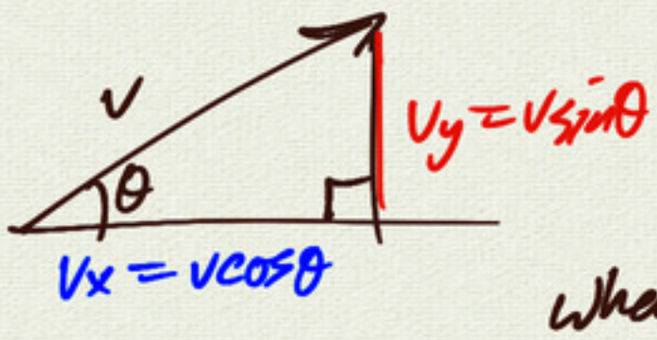
example:



projectile: v const

θ changes

maximize distance d



$x(t) = v_x t$

$y(t) = v_y t - 16t^2$

When do we hit ground? $y(t) = 0$

$v_y t - 16t^2 = 0$

$t(v_y - 16t) = 0$

$t_{end} = \frac{v_y}{16}$

$d = x(t_{end})$

$= v_x\left(\frac{v_y}{16}\right)$

$d = (v\cos\theta)\frac{(v\sin\theta)}{16}$

$\boxed{d(\theta) = \frac{v^2}{16}\sin\theta\cos\theta}$

$d(\theta) = \frac{v^2}{32}\sin 2\theta$ (avoid product rule)

$d(0) = 0$

$d(\frac{\pi}{2}) = 0$

$\sin 2\theta = 2\sin\theta\cos\theta$

$d'(\theta) = \frac{v^2}{32}\cos 2\theta\,(2)$

$= \frac{v^2}{16}\cos 2\theta$

$d''(\theta) = -\frac{v^2}{8}\sin 2\theta$

critical pts: $d'(\theta) = 0$

$\frac{v^2}{16}\cos 2\theta = 0$

$\cos 2\theta = 0$

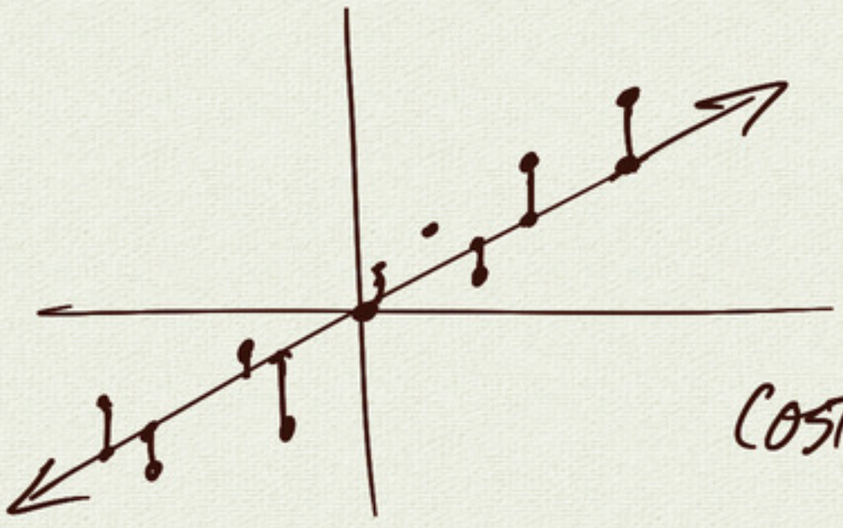
$2\theta = \frac{\pi}{2}$

$\theta = \pi/4$

$d''(\pi/4) = -\frac{v^2}{8}\underbrace{\sin\frac{\pi}{2}}_{1} < 0$

local max

linear regression



data $(x_1, y_1) \dots (x_n, y_n)$

find best fit line (thru origin)

$y = mx$ model

cost $C = \sum_{i=1}^{n} (y_i - mx_i)^2$

minimize cost (choose m)

example: (x_1, y_1) (x_2, y_2) (x_3, y_3)

$$\Rightarrow C(m) = (y_1 - mx_1)^2 + (y_2 - mx_2)^2 + (y_3 - mx_3)^2$$

$$\Rightarrow C'(m) = 2(y_1 - mx_1)(-x_1) + 2(y_2 - mx_2)(-x_2) + 2(y_3 - mx_3)(-x_3)$$

$\frac{d}{dm}(5m) = 5$

$\frac{d}{dm}(m^2) = 2m$

(solve for m)

$$C'(m) = \sum_{i=1}^{n} 2(y_i - mx_i)(-x_i)$$

$$C'(m) = 0 \Rightarrow \sum_{i=1}^{n} (y_i - mx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} (-x_i y_i + m x_i^2) = 0$$

$$\Rightarrow m = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$\sum_{i=1}^{n} x_i y_i$ ← looks like $\vec{x} \cdot \vec{y}$

$\sum_{i=1}^{n} x_i^2$ ← looks like $|\vec{x}|^2$