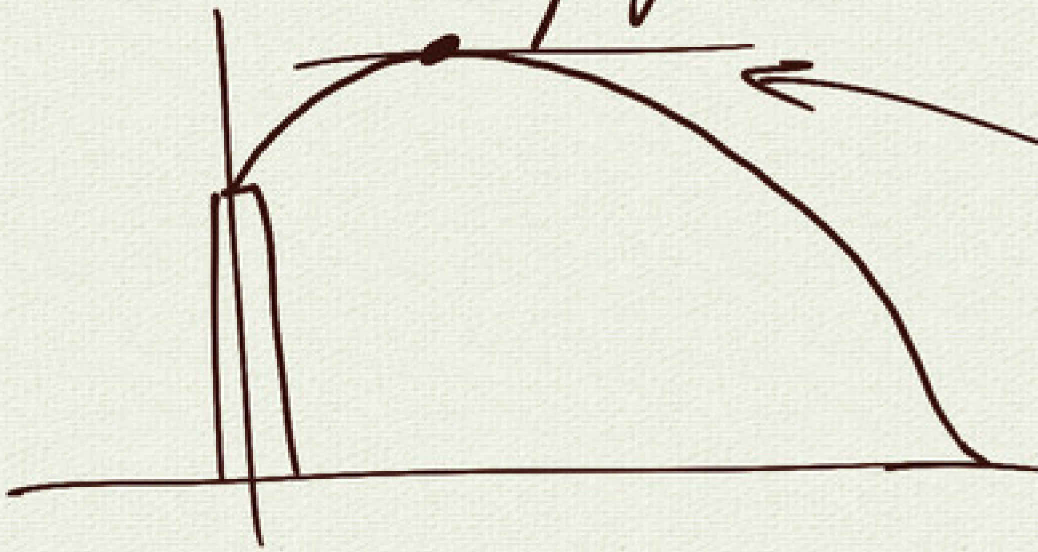
10.4 Optimization

← find min/max

projectile motion



← find max height

maximize $y(t)$

$\Longrightarrow$ find critical pts (where $y'(t)=0$)

$$y(t) = y_0 + v_y t - 16t^2$$

$$\Longrightarrow y'(t) = v_y - 32t$$

$$y''(t) = -32$$

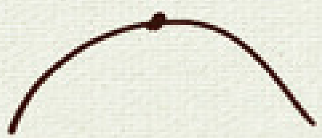
critical pt:

$$y'(t) = 0$$

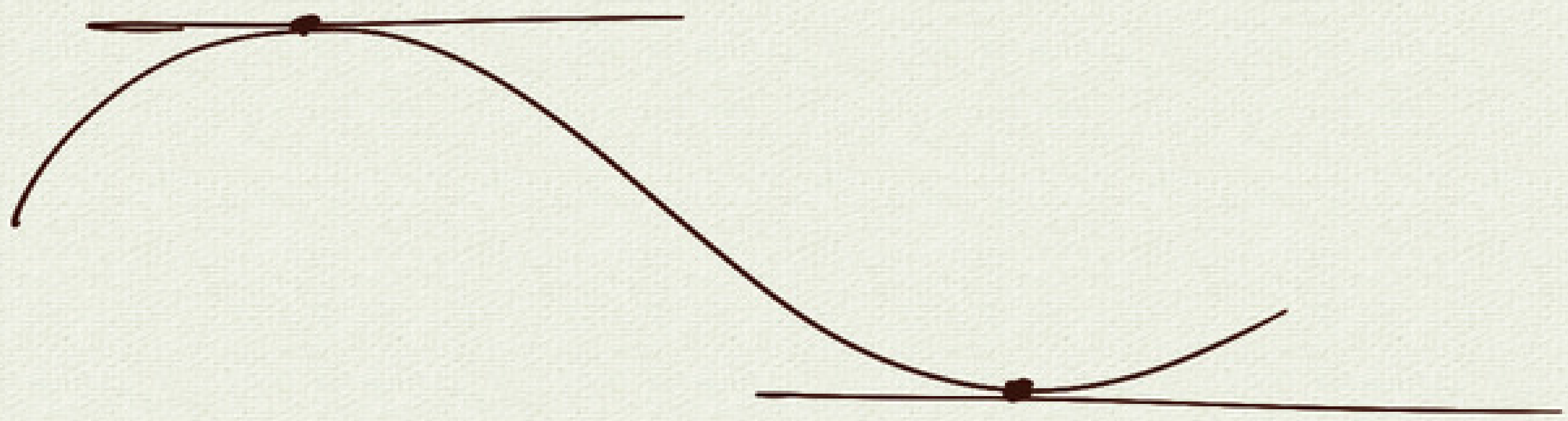
$$v_y = 32t$$

$$t_{max} = \frac{v_y}{32}$$

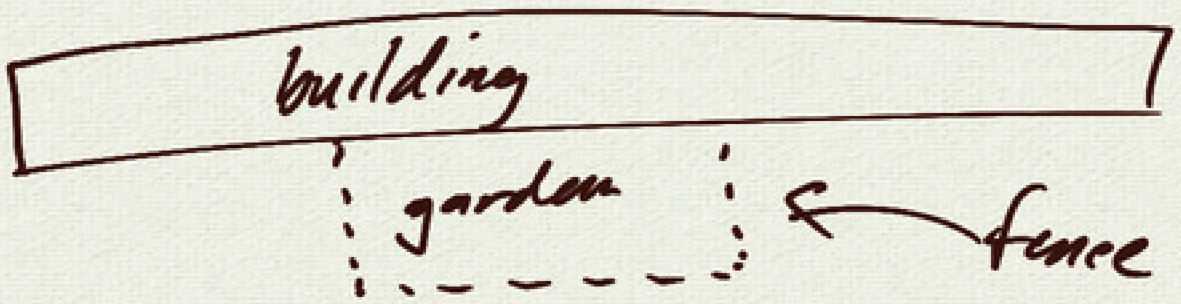
$$y''(t_{max}) = -32 < 0$$



local max

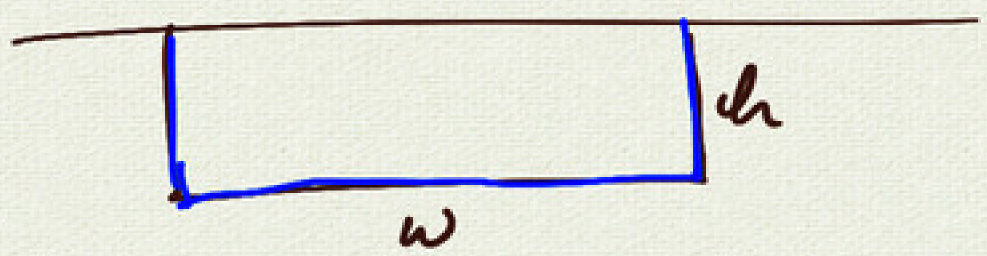


example:



100m fencing material

maximize area of garden (assume rectangle)



$\underset{\text{area}}{A} = wh$

constraint: $100 = 2h + w$

$\Rightarrow w = 100 - 2h$

$\Rightarrow A = wh$

$\boxed{A(h) = (100 - 2h) \cdot h}$ maximize

$A(h) = 100h - 2h^2$

$A'(h) = 100 - 4h$

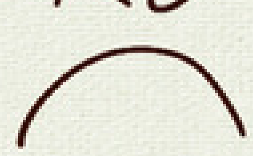
critical pts: $A'(h) = 0$

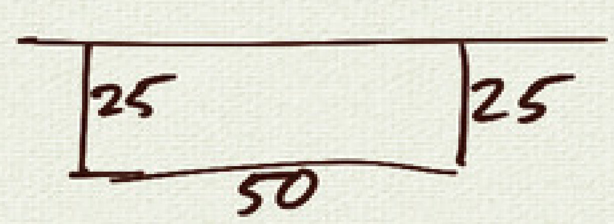
$100 - 4h = 0$

$h = 25 \rightarrow w = 50$

$A''(h) = -4$

$A''(25) = -4 < 0$ local max

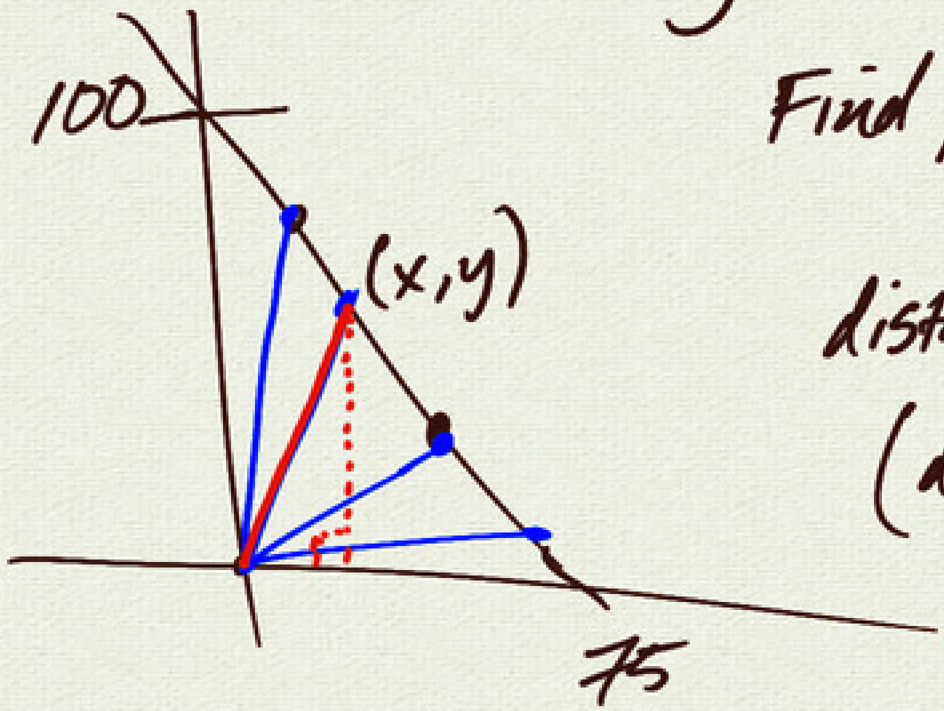




example 2

line

$4x + 3y = 300$



Find point on line closest to origin

distance $d = \sqrt{x^2 + y^2}$

(distance formula $d = \sqrt{(x-0)^2 + (y-0)^2}$, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$)

function: $f(x) = x^2 + y^2$ ← $4x + 3y = 300$, $y = \frac{1}{3}(300 - 4x)$

$f(x) = x^2 + \left[\frac{1}{3}(300 - 4x)\right]^2$

$$f'(x) = 2x + \frac{2}{3^2}(300 - 4x)(-4)$$
$$= 2x - \frac{8}{3^2} \cdot 300 + \frac{32}{3^2}x$$
$$= 2x + \frac{32}{9}x - \frac{800}{3}$$
$$= \frac{18 + 32}{9}x - \frac{800}{3}$$
$$= \frac{50}{9}x - \frac{800}{3}$$

critical pts:

$f'(x) = 0$

$\frac{50}{9}x = \frac{800}{3}$

$50x = 2400$

$x = 48$

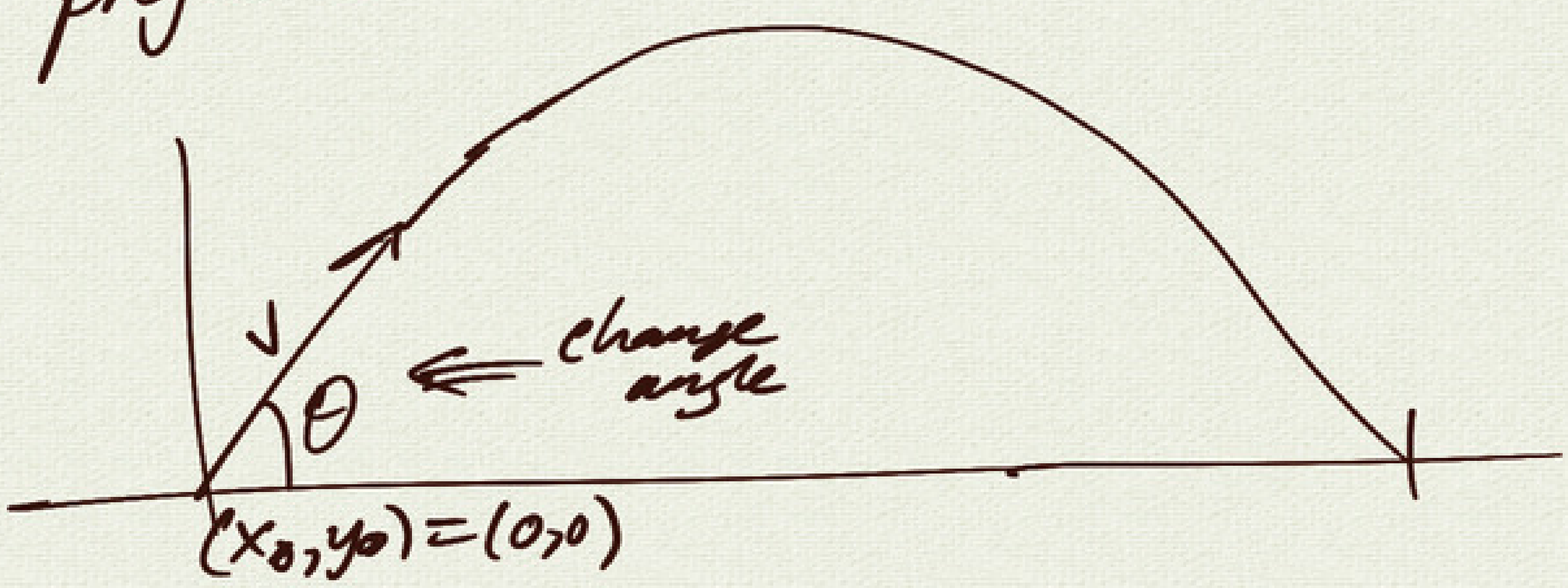
$y = \frac{1}{3}(300 - \underbrace{4 \cdot 48}_{192})$

$= 36$

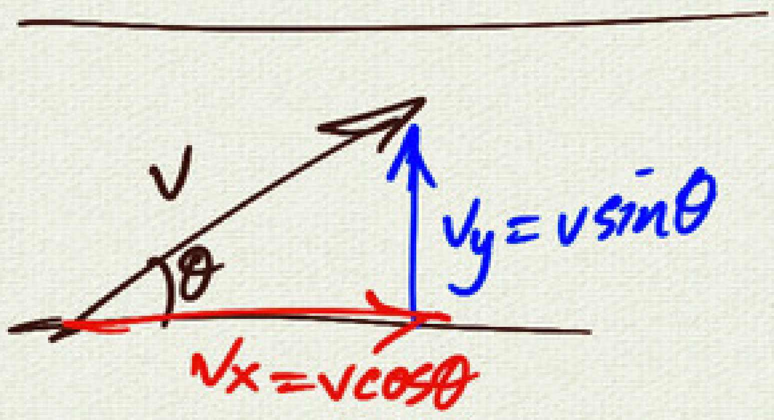
$f''(x) = \frac{50}{9}$

$f''(48) = \frac{50}{9} > 0$ local min

projectile motion



find θ to maximize distance



$x(t) = v_x t$

$y(t) = v_y t - 16t^2$

when do we hit ground?

$y(t) = 0$

$y(t) = t(v_y - 16t) = 0$

$t = 0$ or $t_{ground} = \frac{v_y}{16}$

distance $d = x(t_{ground})$

$= v_x \cdot t_{ground}$

$= v_x \frac{v_y}{16}$

$= \frac{(v\cos\theta)(v\sin\theta)}{16}$

$d(\theta) = \frac{v^2}{16}\cos\theta\sin\theta$

check: $d(\frac{\pi}{2}) = 0$

$d(0) = 0$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\boxed{d(\theta) = \frac{v^2}{32}\sin 2\theta}$$

$d'(\theta) = \frac{v^2}{32}\cos 2\theta \cdot 2$

$= \frac{v^2}{16}\cos 2\theta$

$d''(\theta) = -\frac{v^2}{16}\sin 2\theta$

critical pts: $d'(\theta) = 0$

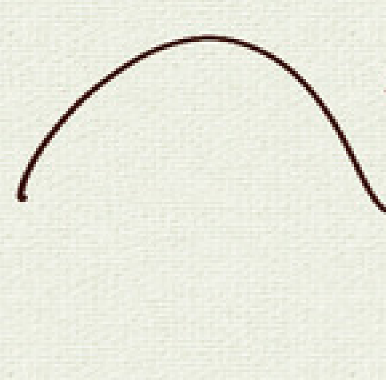
$\frac{v^2}{16}\cos 2\theta = 0$

$\cos 2\theta = 0$

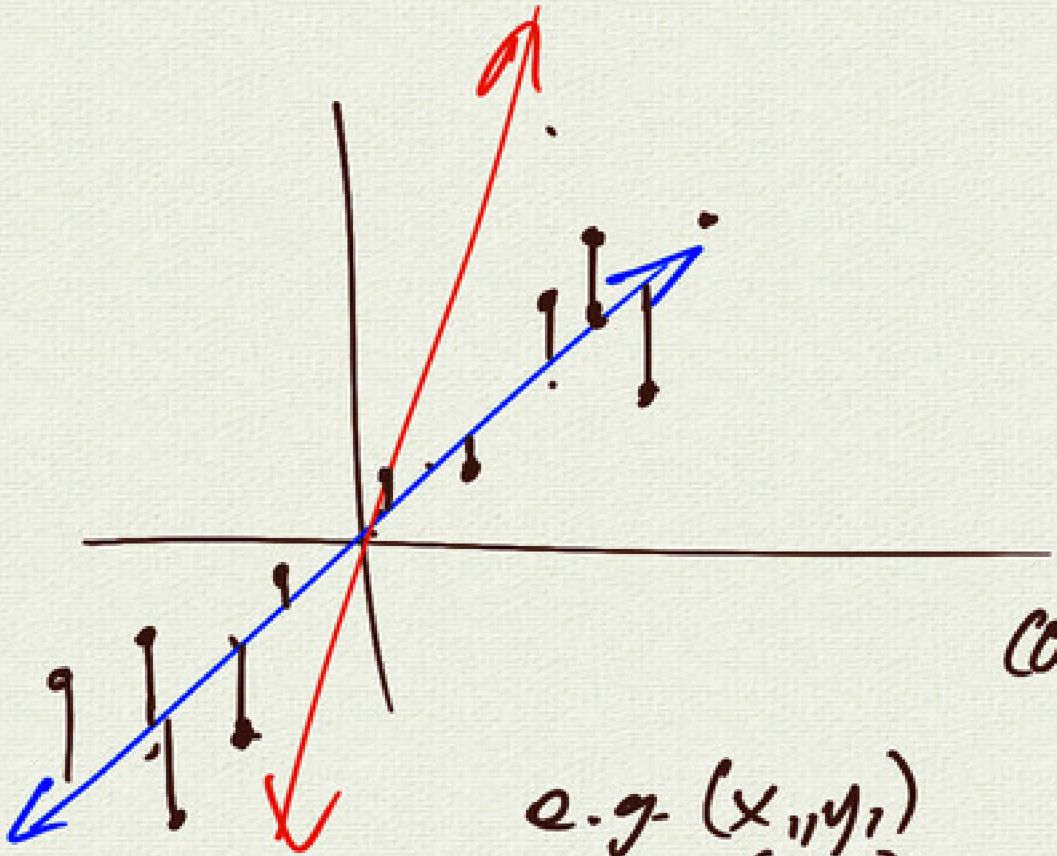
$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$d''(\frac{\pi}{4}) = -\frac{v^2}{16}\underbrace{\sin\frac{\pi}{2}}_{1} < 0$



local max



data $(x_1, y_1) \ldots (x_n, y_n)$

find best fit line thru origin

model $y = mx$

cost $C = \sum_{i=1}^{n} (y_i - mx_i)^2$

e.g. (x_1, y_1)
(x_2, y_2)
(x_3, y_3)

$C = (y_1 - mx_1)^2 + (y_2 - mx_2)^2 + (y_3 - mx_3)^2$

$\frac{d}{dm}(5m) = 5$

minimize cost $C(m) \Rightarrow$ find $C'(m)$

(n=3) $C'(m) = 2(y_1 - mx_1)(-x_1)$
$+ 2(y_2 - mx_2)(-x_1)$
$+ 2(y_3 - mx_3)(-x_3)$

in general:

$$C'(m) = \sum_{i=1}^{n} 2(y_i - mx_i)(-x_i)$$

$$C'(m) = 0 \rightarrow \sum_{i=1}^{n} (-x_i y_i + m x_i^2) = 0$$

$$\Rightarrow m = \frac{\sum_i x_i y_i}{\sum x_i^2}$$

linear regression