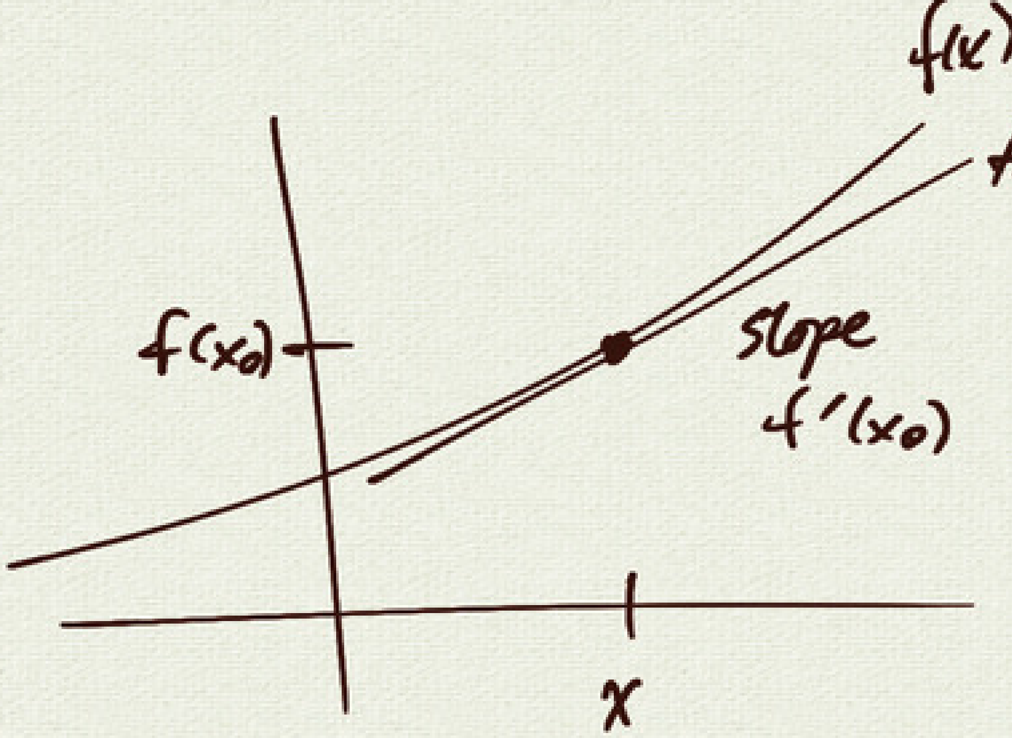
10.5 Linearization



$$\underbrace{\ell(x)=f(x_0)+f'(x_0)(x-x_0)}$$

slope point:

$$y-y_0=m(x-x_0)$$

$$\boxed{y=y_0+m(x-x_0)}$$

idea: near $x=x_0$,

$$f(x)\approx \ell(x)$$

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx}\approx\frac{\Delta y}{\Delta x}$$

$$f'(x_0)\approx\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$f(x_0+\Delta x)-f(x_0)\approx f'(x_0)\Delta x$$

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x \qquad \Delta x=x-x_0$$

$$f(x)\approx\underbrace{f(x_0)+f'(x_0)(x-x_0)}_{\ell(x)}$$

linear approximation

example:

approximate $\sqrt{101}$

near $x=100$:

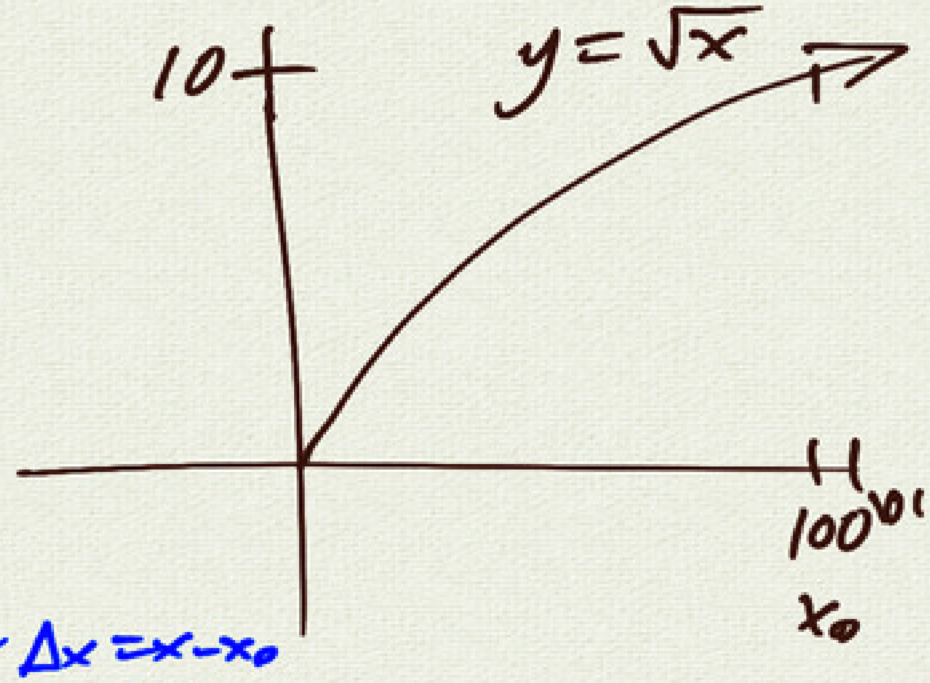
$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

$$\approx f(100) + f'(100)(1)$$

$\Delta x = x - x_0$
$= 1$

$$= 10 + \frac{1}{20}\cdot 1$$

$$= 10.05$$



$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f'(100) = \frac{1}{20}$

calculator: $\sqrt{101} \approx 10.04987$

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example:

approximate $(3.9)^3$

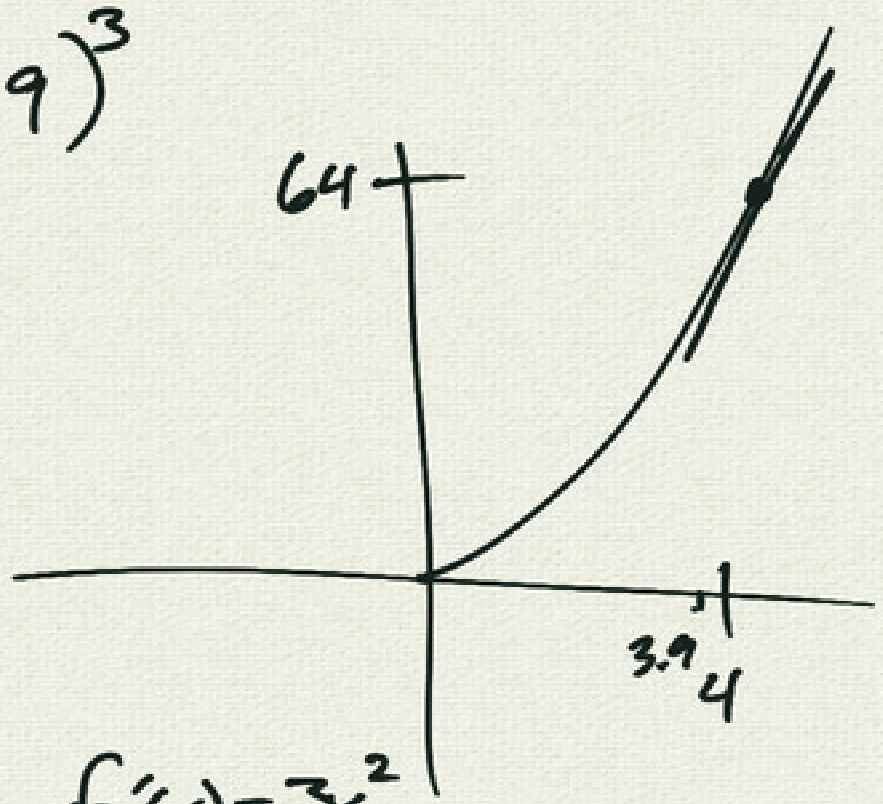
$f(x) = x^3$ near $x=4$

$$f(x) \approx f(x_0) + f'(x_0)\underbrace{\overset{3.9-4}{(x-x_0)}}_{\Delta x}$$

$$= 64 + 48(-.1)$$

$$= 64 - 4.8$$

$$= 59.2$$



$f'(x) = 3x^2$

$f'(4) = 48$

calculator: $(3.9)^3 \approx 59.3$

example: approximate $\sin(.01)$

$f(x) = \sin x$ approximate near $x = 0$ ← x_0

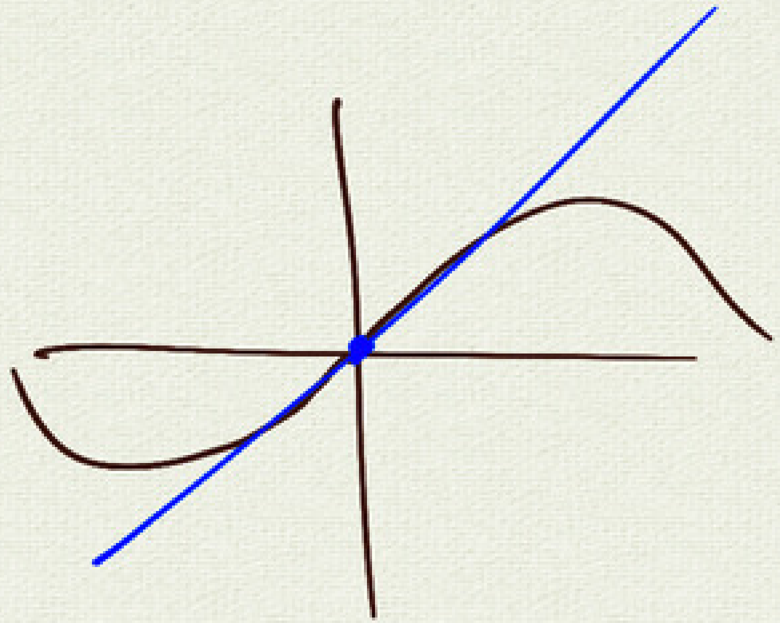
$$f(x) \approx f(0) + f'(0)(x - 0)$$

$$= \underbrace{\sin 0}_{0} + \underbrace{\cos(0)}_{1} x$$

$\sin x \approx x$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$\sin x \approx x$



Babylonians/Greeks

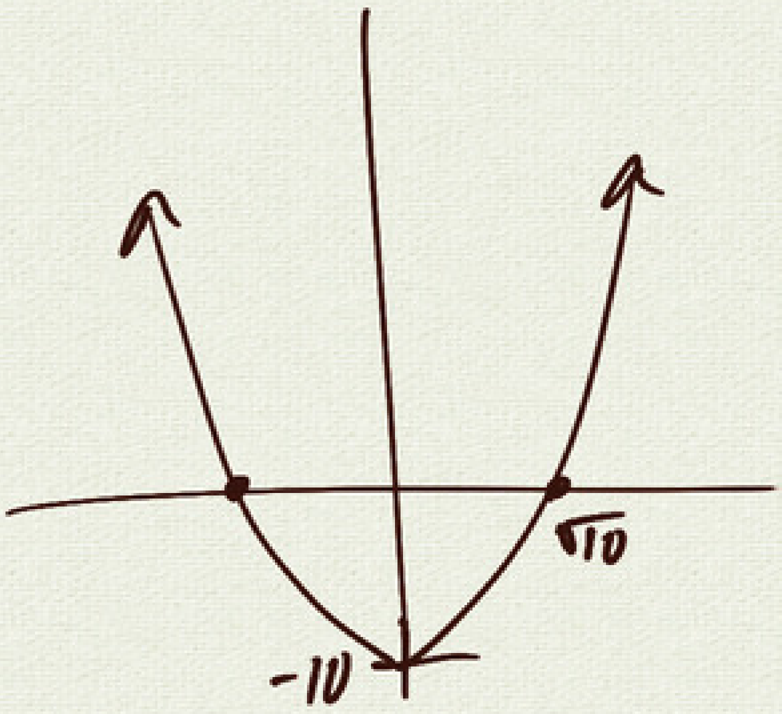
guess:
$x_0 \Longrightarrow$ next guess

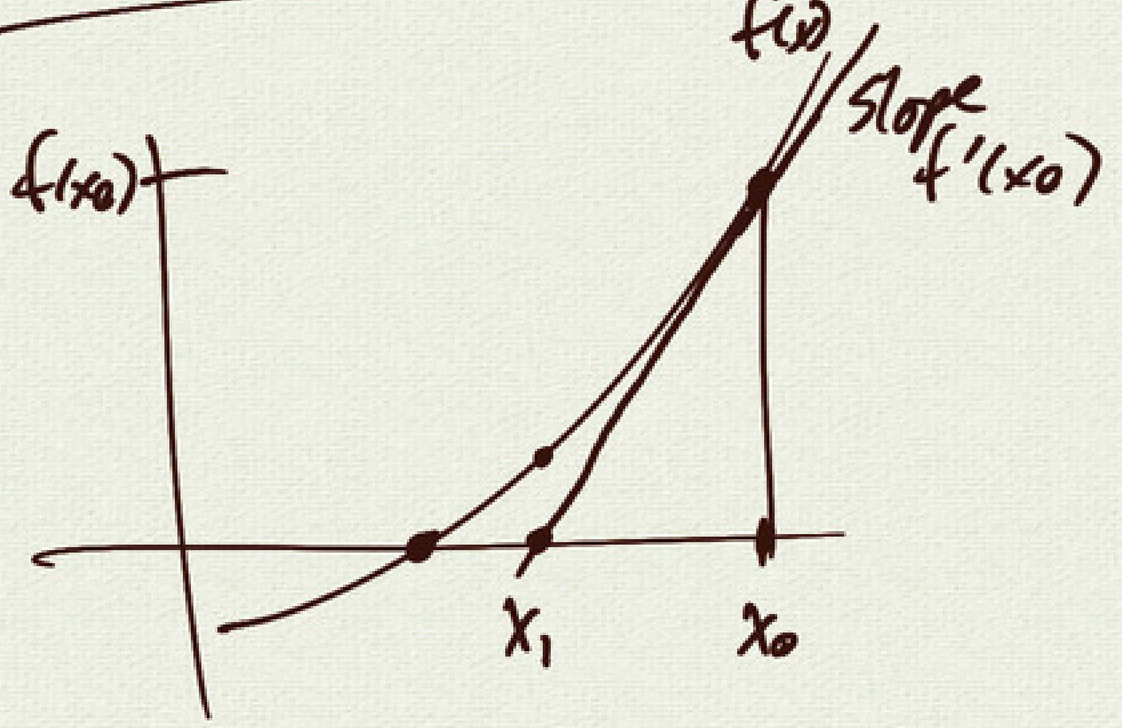
$$x_1 = \frac{x_0 + \frac{10}{x_0}}{2}$$

Newton's Method

find $\sqrt{10}$

$f(x) = x^2 - 10$





$$f'(x_0) = \frac{\text{rise}}{\text{run}} = \frac{f(x_0)}{x_0 - x_1}$$

$$\Longrightarrow x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Newton's Method

$f(x) = x^2 - 10$

$f'(x) = 2x$

$$\Longrightarrow x_1 = x_0 - \frac{(x_0^2 - 10)}{2x_0}$$

$$= \frac{2x_0^2 - (x_0^2 - 10)}{2x_0}$$

$$= \frac{x_0^2 + 10}{2x_0}$$

$$x_1 = \frac{x_0 + 10/x_0}{2}$$