9.6

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$$x+y+z=100$$
$$x+2z=125$$
$$-y+2z=25$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 1 & 0 & 2 & 125 \\ 0 & -1 & 2 & 25 \end{array}\right)$$

$\downarrow$ $-R_1+R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & -1 & 1 & 25 \\ 0 & -1 & 2 & 25 \end{array}\right)$$

$\downarrow$ $-1 R_2$

$$\left(\begin{array}{ccc|c} 1 & \boxed{1} & 1 & 100 \\ 0 & 1 & -1 & -25 \\ 0 & \boxed{-1} & 2 & 25 \end{array}\right)$$

$\downarrow$ $-R_2+R_1$ , $R_2+R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{2} & 125 \\ 0 & 1 & \boxed{-1} & -25 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\downarrow$ $-2R_3+R_1$
$1 \cdot R_3+R_2$

...

row operations:

$k \cdot R_i$

$k \cdot R_i + R_j$

$R_{ij}$ swap

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$$\frac{x-1}{7} + \frac{y-2}{8} + \frac{z-3}{4} = 0$$

$$x + y + z = 6$$

$$\frac{x+2}{3} + 2y + \frac{z-3}{3} = 5$$

.56

$$8(x-1) + 7(y-2) + 14(z-3) = 0$$

$$8x - 8 + 7y - 14 + 14z - 42 = 0$$

$$8x + 7y + 14z = 64$$

# 3.7 Matrices

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ matrix: a 2D array of numbers

$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ 2×3 matrix

#rows #columns

notation:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

1st row 3rd column

$a_{ij}$ row column

basic operations:

① addition: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} \boxed{4} & 7 \\ 3 & 10 \end{pmatrix}$

② scalar mult. $3 \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = ?$

not defined

---

matrix multiplication:

$1 \cdot 2 + 3 \cdot 3$

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{11} & \boxed{3} \\ \boxed{16} & 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = ? \begin{pmatrix} \boxed{\phantom{0}} & \end{pmatrix}$

not defined

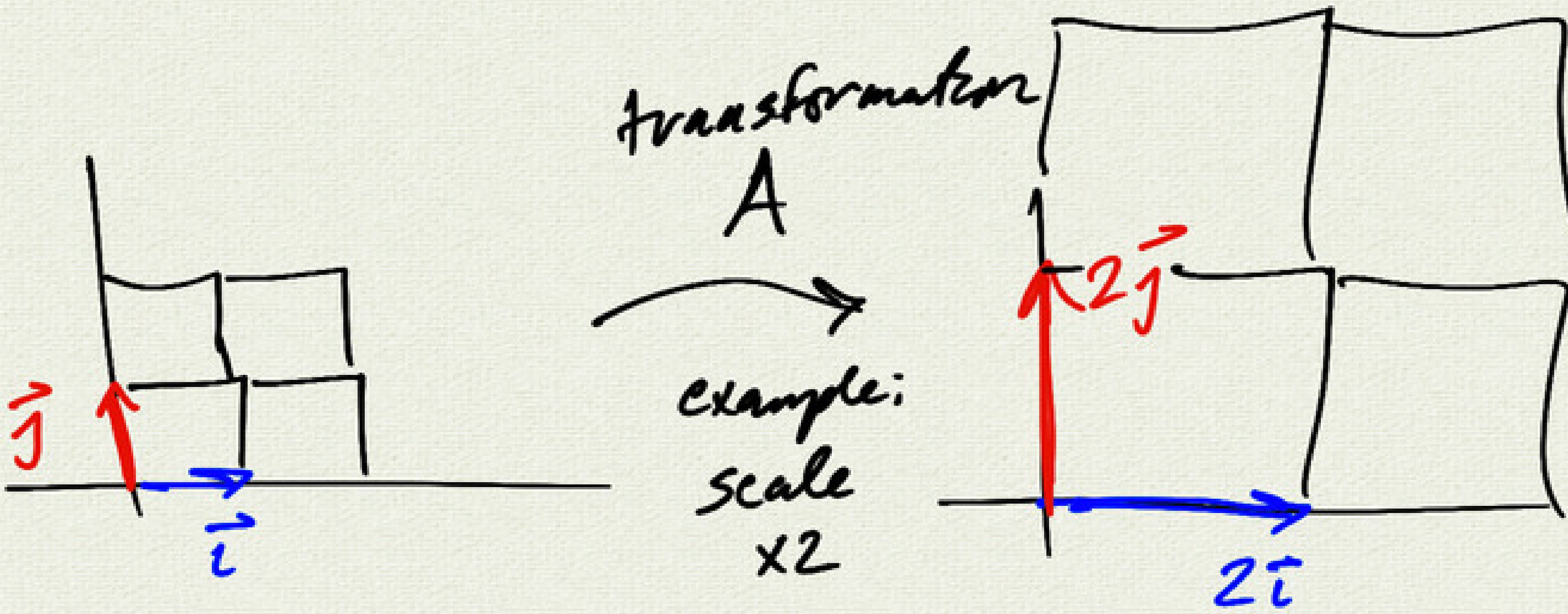
$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 9 & \cdot \\ \boxed{12} & \cdot \\ \boxed{15} & \cdot \end{pmatrix}$

$\underbrace{3 \times 2 \quad 2}_{\text{dot product}} \times 2 \Rightarrow 3 \times 2$

$\underbrace{(m \times n) \quad (n}_{\text{must be same (dot product)}} \times \ell) \rightarrow m \times \ell$

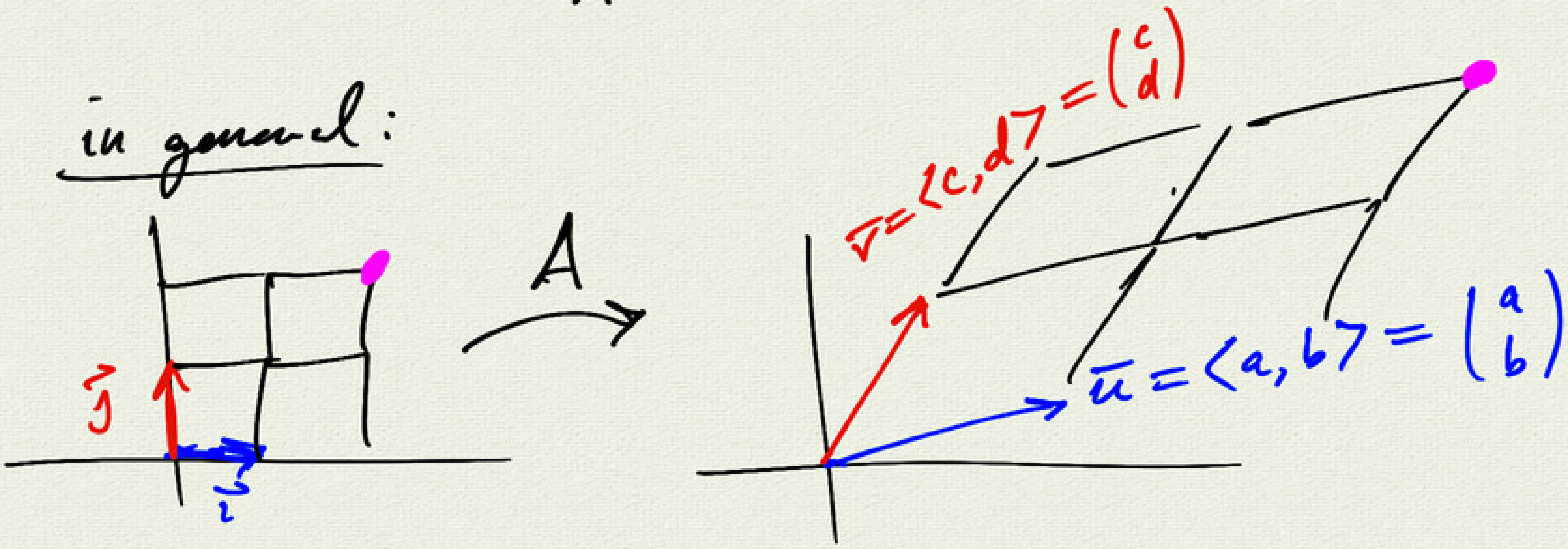
#rows of first matrix

#columns from 2nd matrix




$$\langle 3,4\rangle \xrightarrow{A} \langle 6,8\rangle$$

<u>in general:</u>



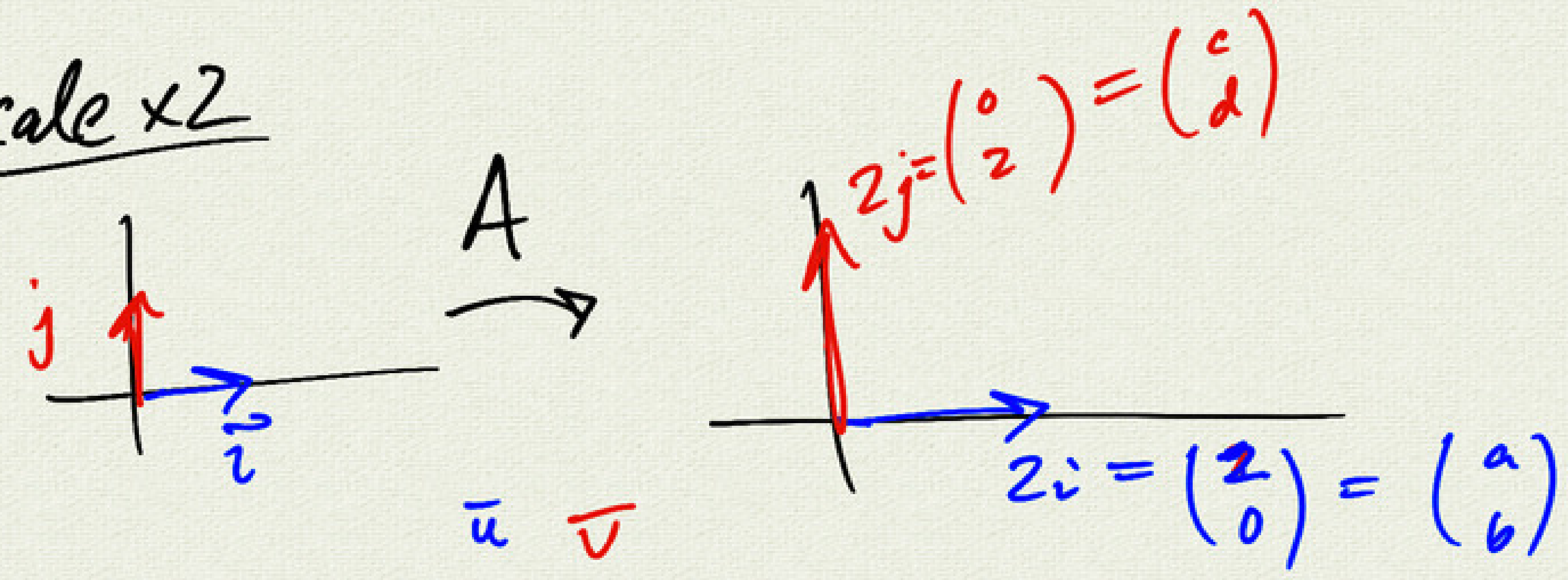

$$\langle x,y\rangle = x\vec{i} + y\vec{j} \xrightarrow{A} x\vec{u} + y\vec{v}$$

$$= x\begin{pmatrix} a \\ b \end{pmatrix} + y\begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$$

scale x2




$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \bar{u} & \bar{v} \\ \end{array}$$

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

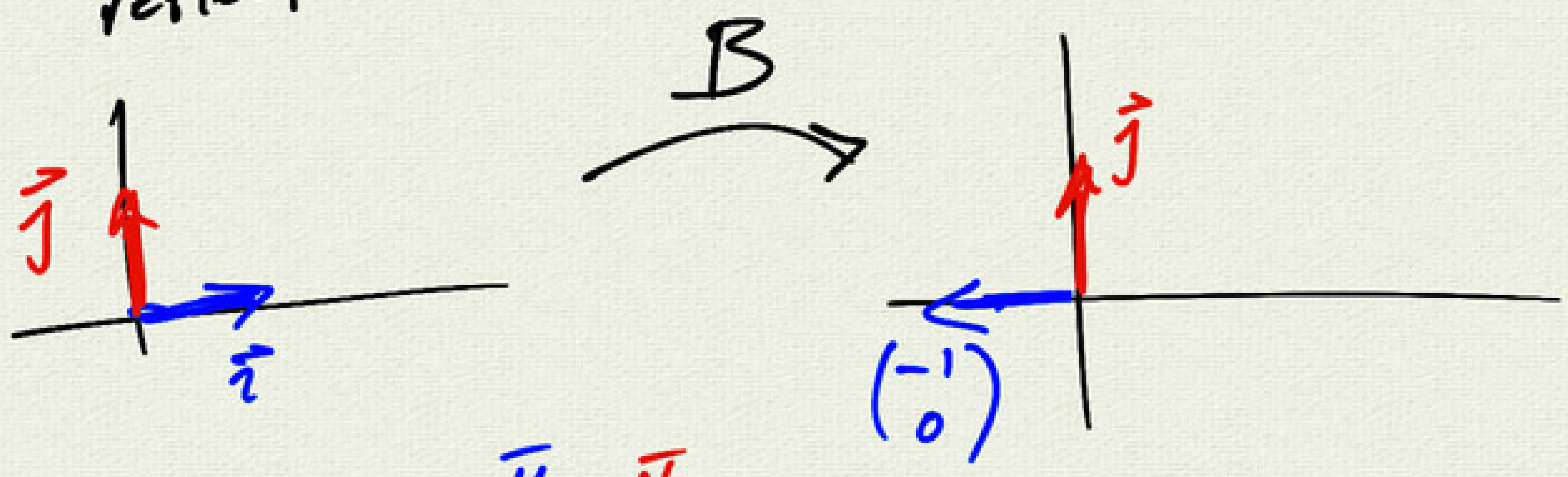
$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

apply A to vector $\langle x, y \rangle = \begin{pmatrix} x \\ y \end{pmatrix}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

---

example 2
reflect across y-axis



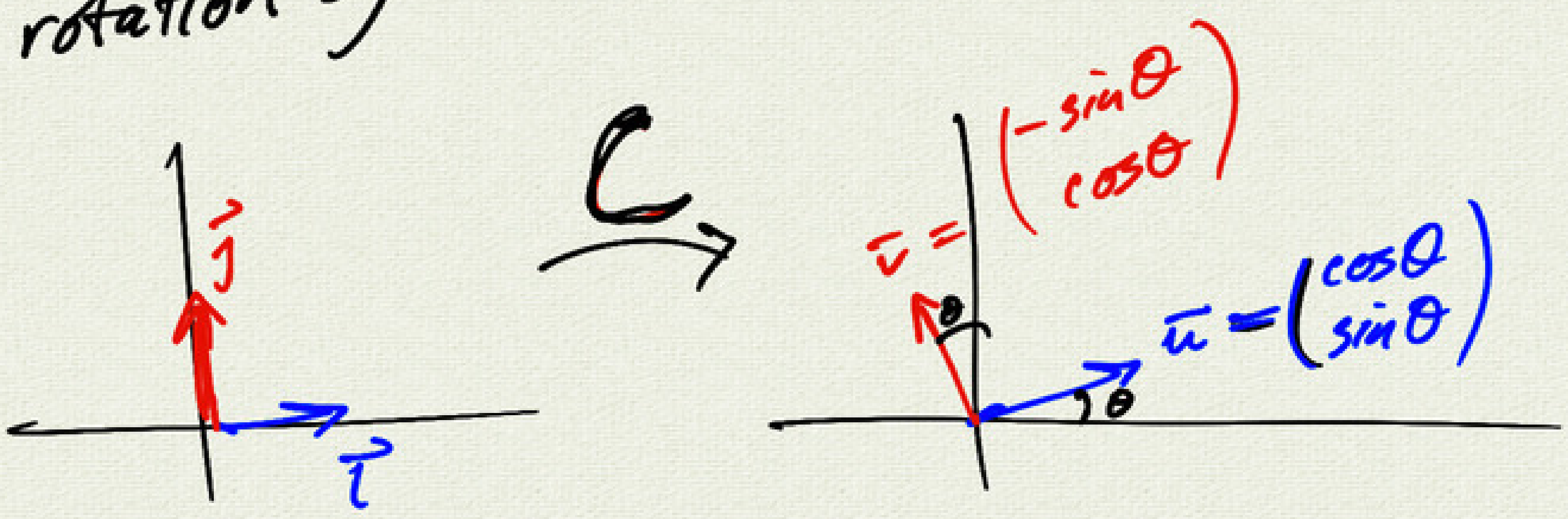

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

apply B to $\begin{pmatrix} x \\ y \end{pmatrix}$:

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

---

example:
rotation by $\theta$




$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

verify $\bar{u} \perp \bar{v}$:

$\bar{u}, \bar{v}$ orthogonal $\iff \bar{u} \cdot \bar{v} = 0$

$\boxed{|\bar{u}||\bar{v}|\cos\theta = 0}$

$|\bar{u}| = 0$ or $|\bar{v}| = 0$

or $\underbrace{\cos\theta = 0}_{\theta = \pi/2}$ perpendicular

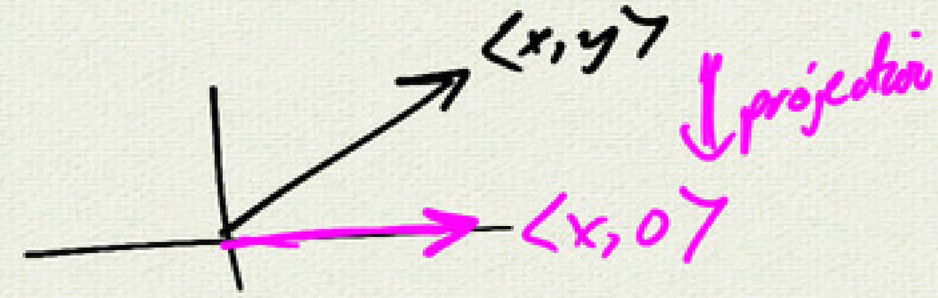
$$\bar{u} \cdot \bar{v} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= -\sin\theta\cos\theta + \sin\theta\cos\theta$$

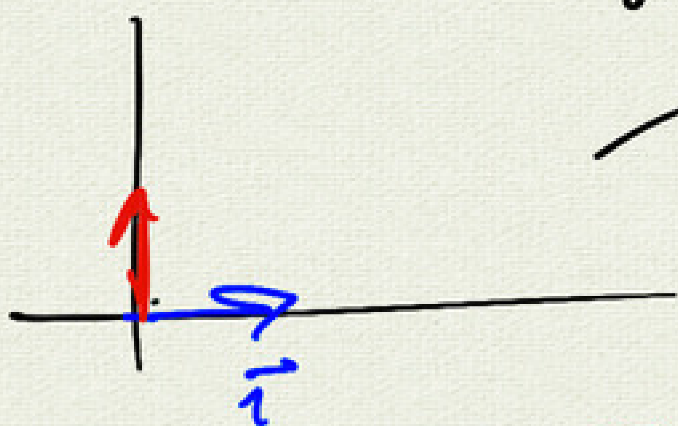
$$= 0 \quad \checkmark$$

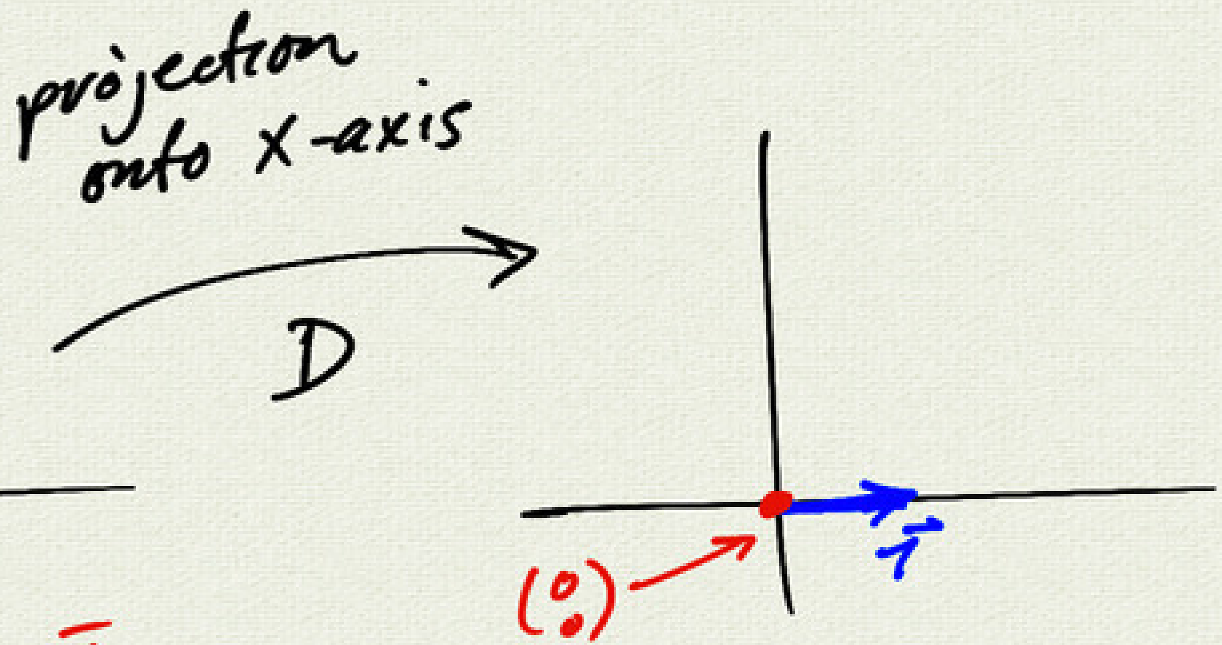
dot product

$$\langle \cos\theta, \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle$$




## example 4




$$D = \begin{array}{c} \begin{array}{cc} \bar{u} & \bar{v} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

apply $D$ to $\binom{x}{y}$:

$$D\binom{x}{y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\binom{x}{y} = \binom{x}{0}$$

---

matrix $\longleftrightarrow$ linear transformation

multiplication $\longleftrightarrow$ apply the transformation