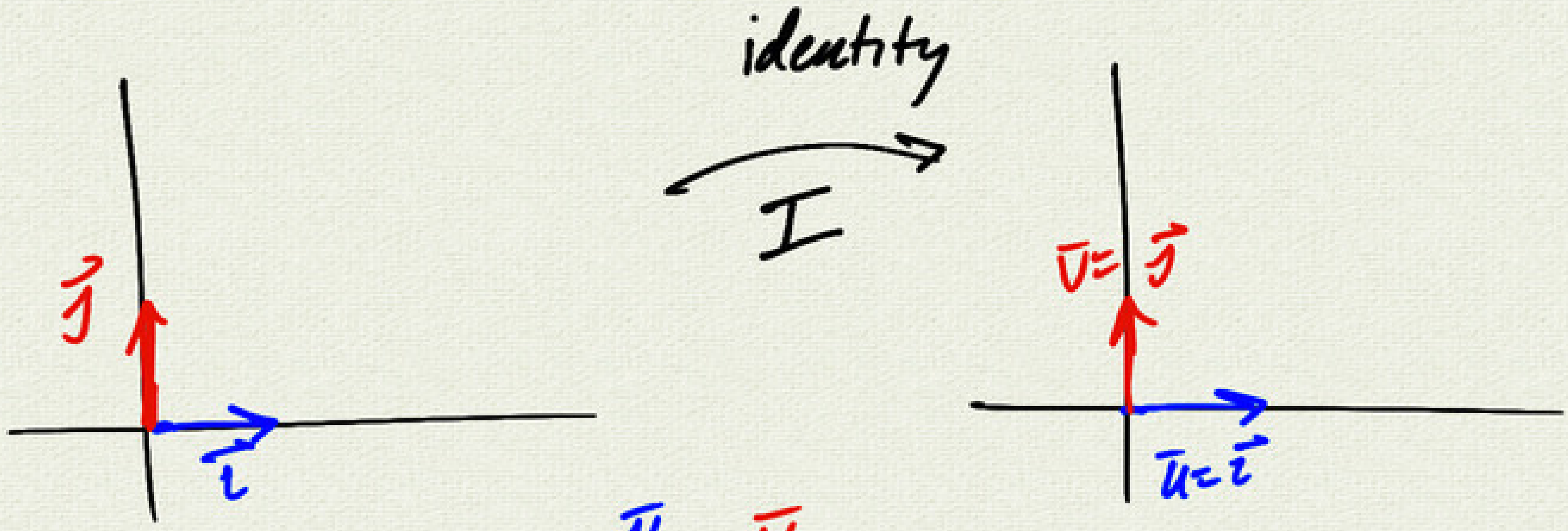
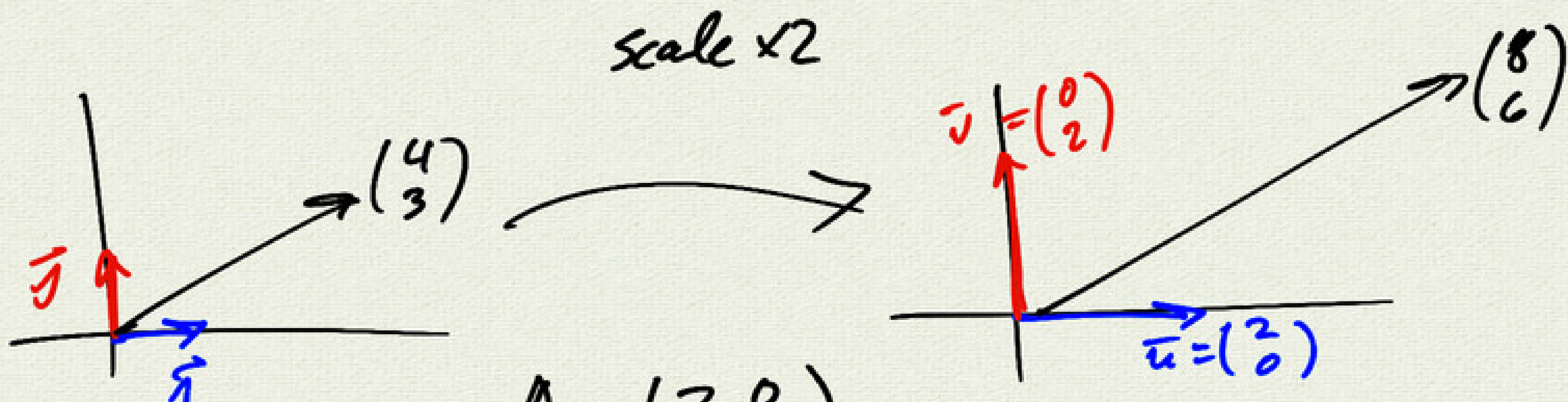
3.8 Matrix Inverses



$$I = \begin{array}{c} \begin{array}{cc} \bar{u} & \bar{v} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$I\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \boxed{1 \quad 0} \\ 0 \quad 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$$

scale $\times \frac{1}{2}$

$$B = \tfrac{1}{2}I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$AB = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_{(2I)}\underbrace{\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}}_{(\frac{1}{2}I)} = I$$

also: $BA = I$

A, B are <u>inverses</u>

notation:

$$B = A^{-1}$$

$$\underset{A}{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (ad-bc)\, I$$

let $B = \dfrac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

then $AB = I$, i.e. $B = A^{-1}$

ok if $ad-bc \neq 0$

determinant

Summary

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\det A = ad - bc$ (notation: $|A|$
$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$)

A^{-1} exists $\Longleftrightarrow \det A \neq 0$

$$\left(A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \right)$$

linear system:

$$3x + 2y = 1$$
$$2x + y = 1$$

equivalent

matrix version

$$\underbrace{\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underbrace{A^{-1} A}_{I} \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \Longrightarrow A^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$\det A = -1$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

solution $\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

check:

$$3x + 2y = 1$$
$$2x + y = 1 \quad \checkmark$$

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even more:

$$3x + 2y = 4$$
$$2x + y = 5$$

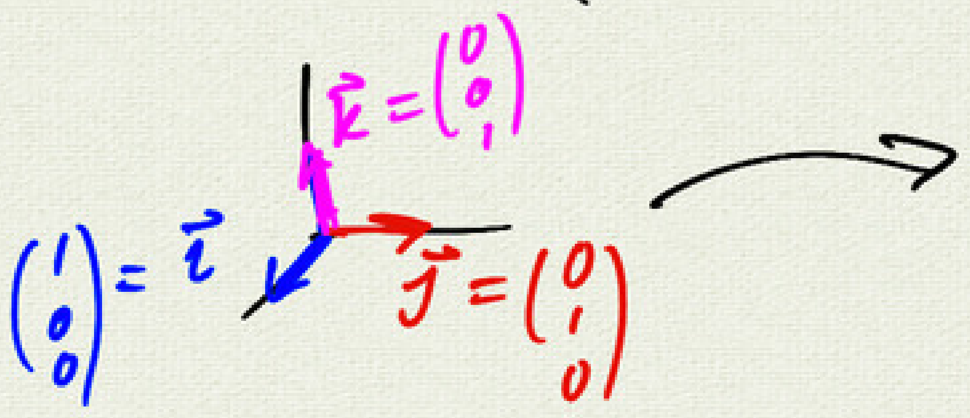
$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \qquad A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$

$\mathbb{R}^3$

3×3 $\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$



<u>example:</u>

$$3x + 6y = 15$$
$$y - z = -1$$
$$-2x - 4y + z = -7$$

$$\underbrace{\begin{pmatrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

find A^{-1}:

$$\left(\underbrace{\begin{matrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \right) \longrightarrow \left(I \mid A^{-1} \right)$$

$\downarrow \frac{1}{3}R_1$

$$\left(\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{matrix} \middle| \begin{matrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right)$$

$\downarrow +2R_1 + R_3$

$$\left(\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 2/3 & 0 & 1 \end{matrix} \right)$$

$\downarrow -2R_2 + R_1$

$$\left(\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1/3 & -2 & 0 \\ 0 & 1 & 0 \\ 2/3 & 0 & 1 \end{matrix} \right)$$

$\downarrow -2R_3 + R_1 \ , \quad R_3 + R_2$

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} -1 & -2 & -2 \\ 2/3 & 1 & 1 \\ 2/3 & 0 & 1 \end{matrix}}_{A^{-1}} \right)$$

$$\Rightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

$$\underbrace{A^{-1} A}_{I} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -2 & -2 \\ 2/3 & 1 & 1 \\ 2/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -15 + 2 + 14 \\ 10 - 1 - 7 \\ 10 + 0 - 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

A^{-1} exists $\Longleftrightarrow$ $\det A \neq 0$

3×3 determinant?

$$A = \begin{pmatrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\Rightarrow \det A = 3\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix} - 6\begin{vmatrix} 0 & -1 \\ -2 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ -2 & -4 \end{vmatrix}$$

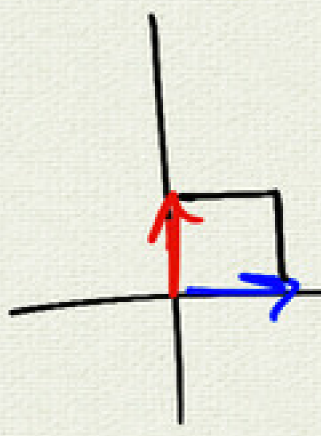
$$= 3\underbrace{(-3)}_{ad-bc} - 6\underbrace{(-2)}_{ad-bc} + 0(\quad)$$

$$= -9 + 12$$

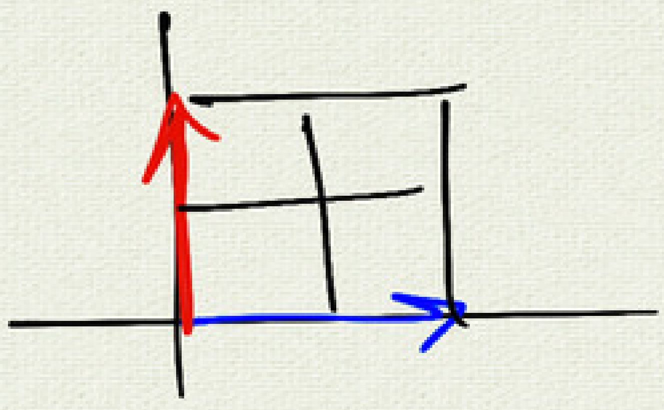
$$= 3$$

examples

scale ×2



$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$



$A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$\underbrace{\det A = 4}_{\text{area scaling}}$ $\quad$ $\det A^{-1} = \frac{1}{4}$

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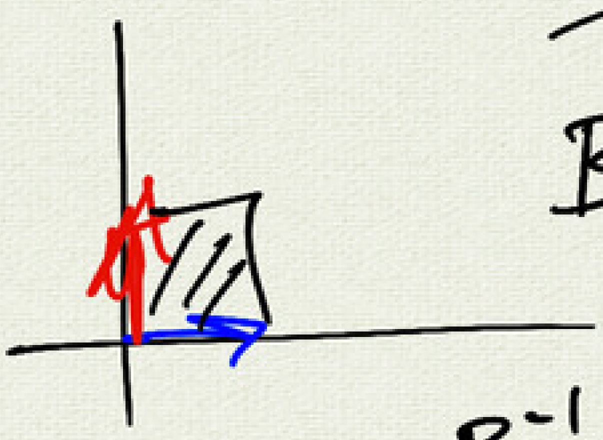
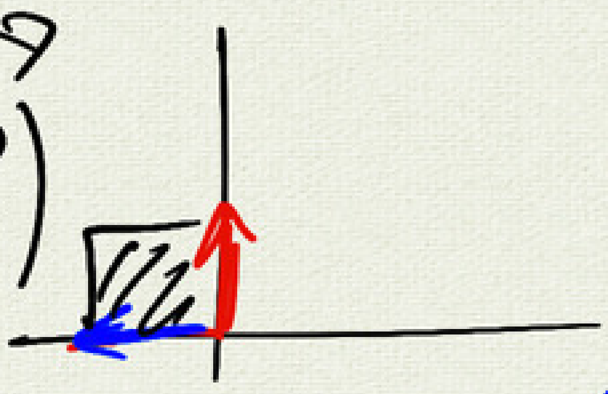
reflection over y-axis



$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



$B^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $BB^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$\det B = -1$

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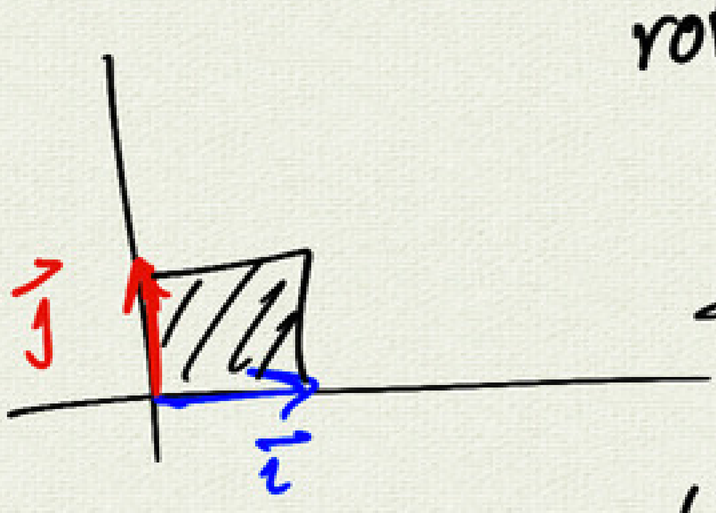
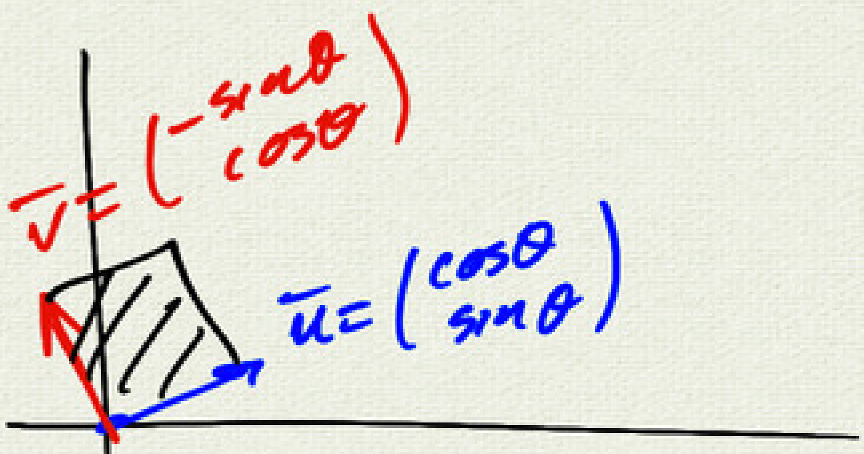
rotation by θ





$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

C^{-1} = rotation by $-\theta$

$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\det C = \cos^2\theta + \sin^2\theta = 1$

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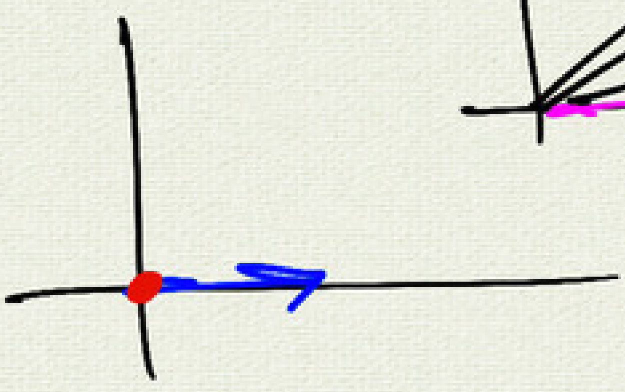
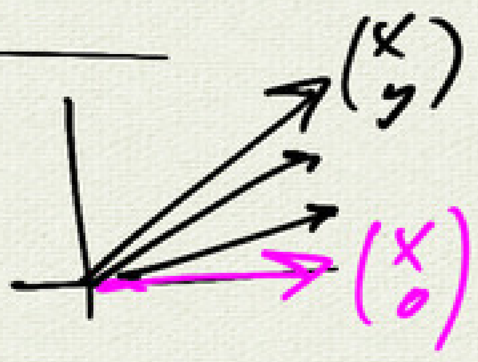
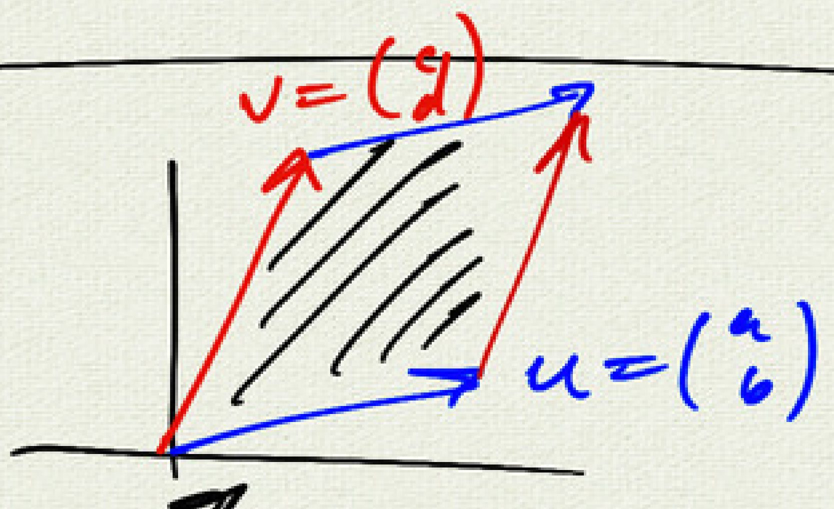
projection onto x-axis





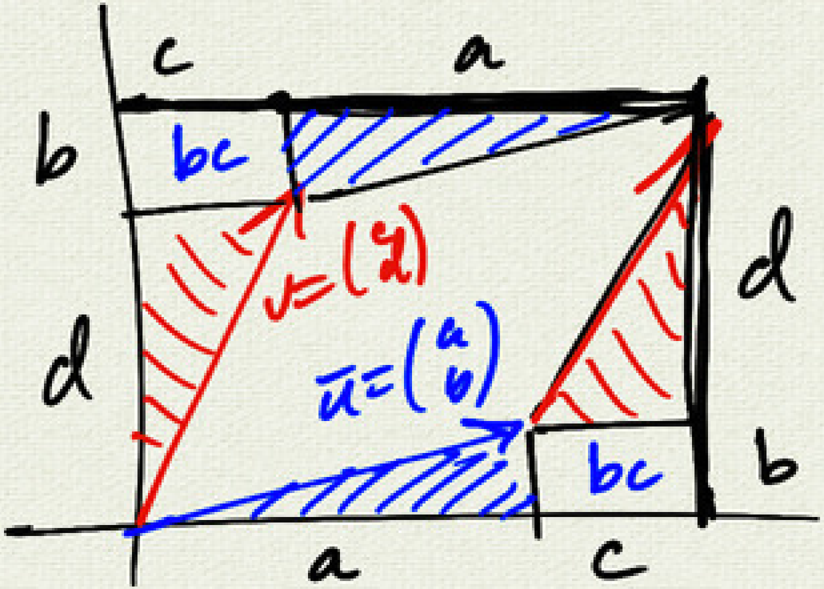
$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

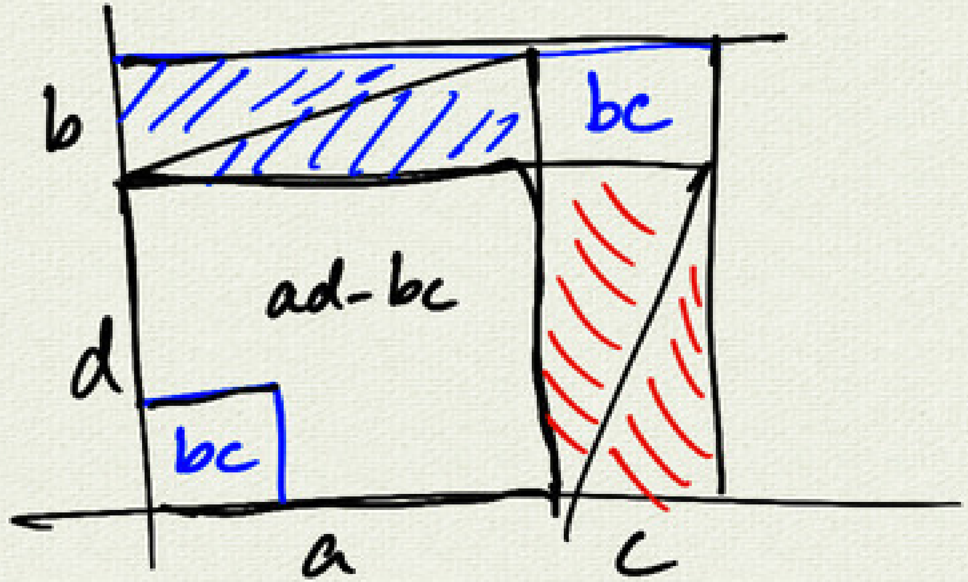
$\det D = 0$



area ▱ $= \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$

3D : parallelopiped





area ▱ = $ad-bc$

algebraic

$(a+c)(b+d)$