3.8 Matrix Inverses

$$I = \begin{array}{c} \begin{array}{cc} \bar{u} & \bar{v} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \quad \text{identity matrix}$$

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$I\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 3 \\ 2 & 4 \end{pmatrix} \quad \left(= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} I \right)$$

---

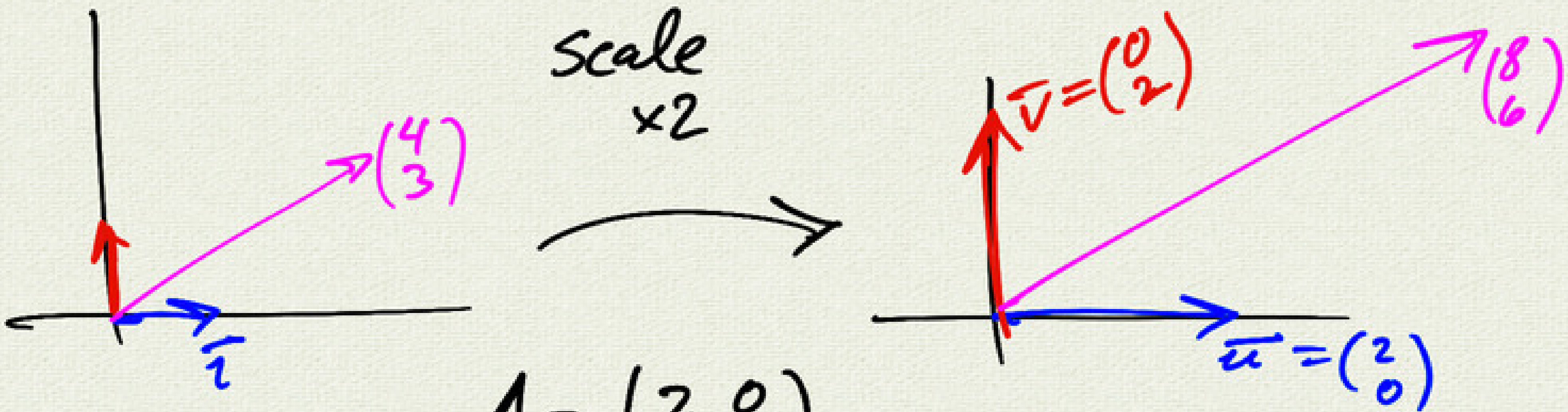
in $\mathbb{R}^3$: $\quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad$ 3×3 identity

---

play: / practice

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc)I$$



$$A=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$$=2I$$

scale $\times\frac{1}{2}$

$$B=\tfrac{1}{2}I=\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}$$

$$AB=\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}$$
$$=2I\left(\tfrac{1}{2}I\right)$$
$$=I$$
also $BA=I$

A, B are <u>inverse</u> matrices | $AB=I=BA$

notation: $B=A^{-1}$

---

back to: $$\underbrace{\begin{pmatrix}a&c\\b&d\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}d&-c\\-b&a\end{pmatrix}}=\begin{pmatrix}ad-bc&0\\0&ad-bc\end{pmatrix}$$
$$=(ad-bc)\,I$$

$\Longrightarrow$ let $B=\dfrac{1}{ad-bc}\begin{pmatrix}d&-c\\-b&a\end{pmatrix}$

then $AB=I=BA$

i.e. $A^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-c\\-b&a\end{pmatrix}$

$A=\begin{pmatrix}1&1\\1&1\end{pmatrix}\longrightarrow ad-bc=0$ A^{-1} undefined

---

<u>summary:</u> A^{-1} exists $\Longleftrightarrow \underbrace{ad-bc}\neq 0$

$\det A$ determinant

$|A|$

$\begin{vmatrix}a&c\\b&d\end{vmatrix}$

$$\left(A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-c\\-b&a\end{pmatrix}\right)$$

example:

$$3x + 2y = 1$$
$$2x + y = 1$$

system of linear equations

$$\underbrace{\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

matrix equation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

if A^{-1} exists,

$$\underbrace{A^{-1} A}_{I} \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \longrightarrow A^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

check: $3x + 2y = 1$
$2x + y = 1$ ✓

$$\begin{aligned} 3x + 2y &= 4 \\ 2x + y &= 5 \end{aligned} \longrightarrow A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$ solution

3x3 case:

$$A^{-1} \text{ exists} \iff \det A \neq 0$$

example from before:

$$3x + 6y = 15$$
$$y - z = -1$$
$$-2x - 4y + z = -7$$

$$\underbrace{\begin{pmatrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix} \quad \text{solution}$$

$$\det A = \begin{vmatrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix} - 6 \begin{vmatrix} 0 & -1 \\ -2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ -2 & -4 \end{vmatrix}$$

bc

ad

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

why?

$$= 3(-3) - 6(-2) + 0(2)$$
$$= -9 + 12$$
$$= 3$$

$$\det A = 3 \neq 0 \Longrightarrow A^{-1} \text{ exists}$$

$$A = \begin{pmatrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{pmatrix} \Rightarrow A^{-1} = ?$$

$$\left(\underbrace{\begin{matrix} 3 & 6 & 0 \\ 0 & 1 & -1 \\ -2 & -4 & 1 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) \longrightarrow \left(I \mid A^{-1} \right)$$

$$\downarrow \tfrac{1}{3}R_1$$

$$\left(\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ \boxed{-2} & -4 & 1 \end{matrix} \;\middle|\; \begin{matrix} \boxed{1/3 \quad 0 \quad 0} \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 1 \end{matrix}\right)$$

$$\downarrow 2R_1 + R_3$$

$$\left(\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ +2/3 & 0 & 1 \end{matrix}\right)$$

$$\downarrow -2R_2 + R_1$$

$$\left(\begin{matrix} 1 & 0 & \boxed{2} \\ 0 & 1 & \boxed{-1} \\ 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1/3 & -2 & 0 \\ 0 & 1 & 0 \\ 2/3 & 0 & 1 \end{matrix}\right)$$

$$\downarrow -2R_3 + R_1 , \quad R_3 + R_2$$

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} -1 & -2 & -2 \\ 2/3 & 1 & 1 \\ 2/3 & 0 & 1 \end{matrix}}_{A^{-1}}\right)$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

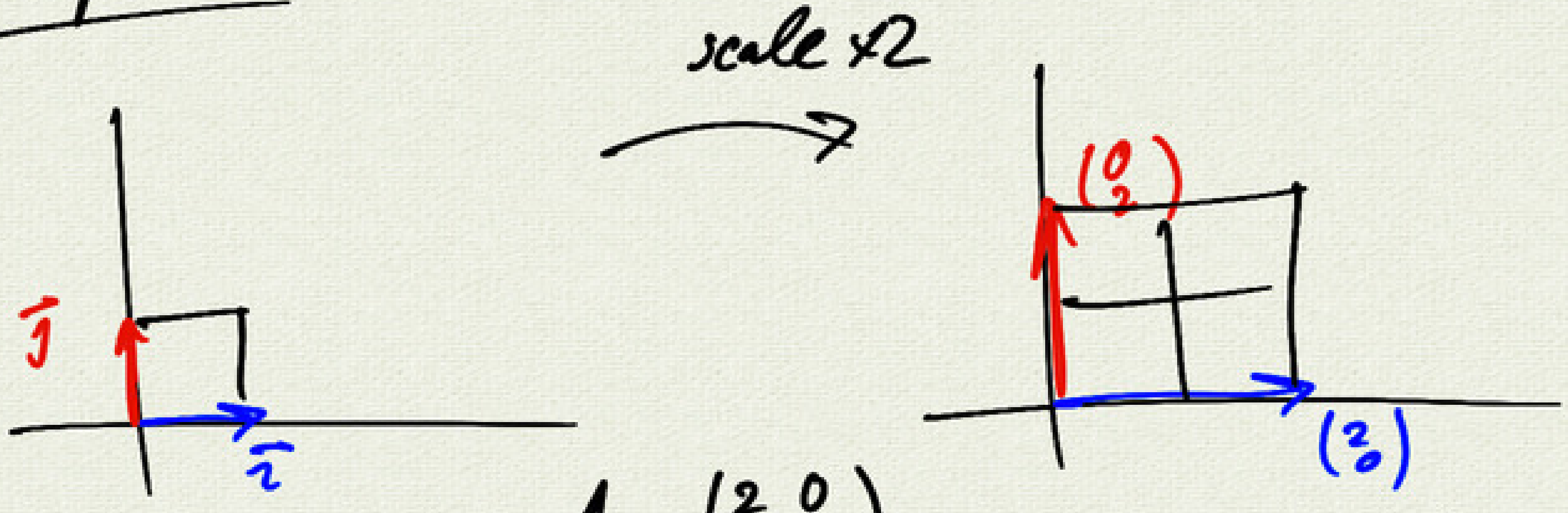
$$\rightarrow \underbrace{A^{-1}A}_{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 & -2 \\ 2/3 & 1 & 1 \\ 2/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ -1 \\ -7 \end{pmatrix}$$

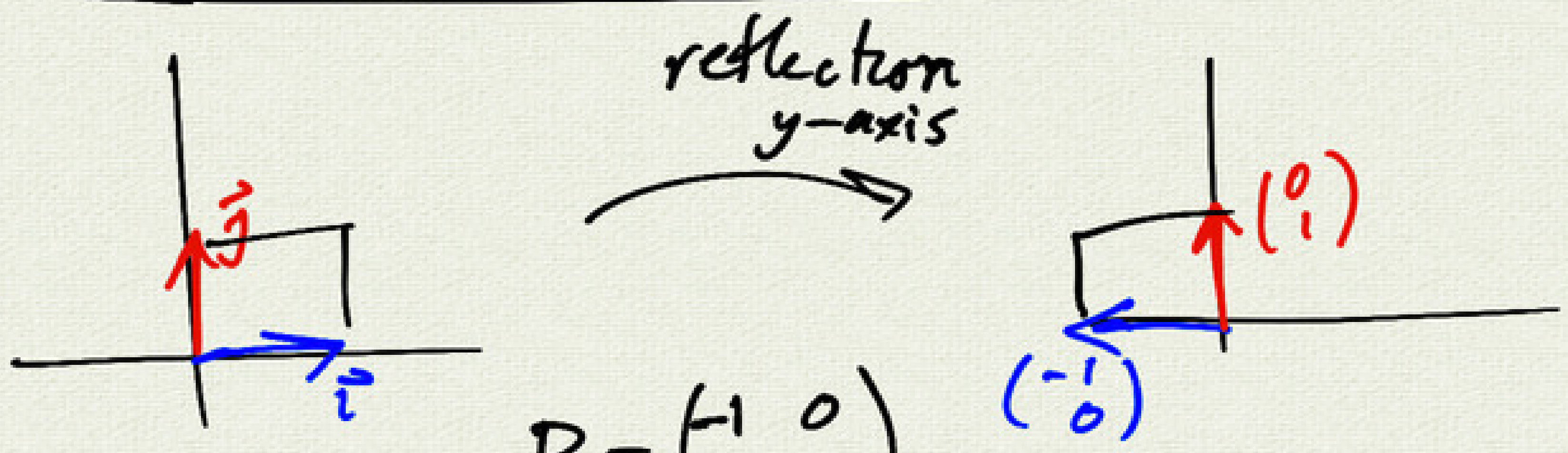
$$= \begin{pmatrix} -15+2+14 \\ 10-1-7 \\ 10+0-7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

examples



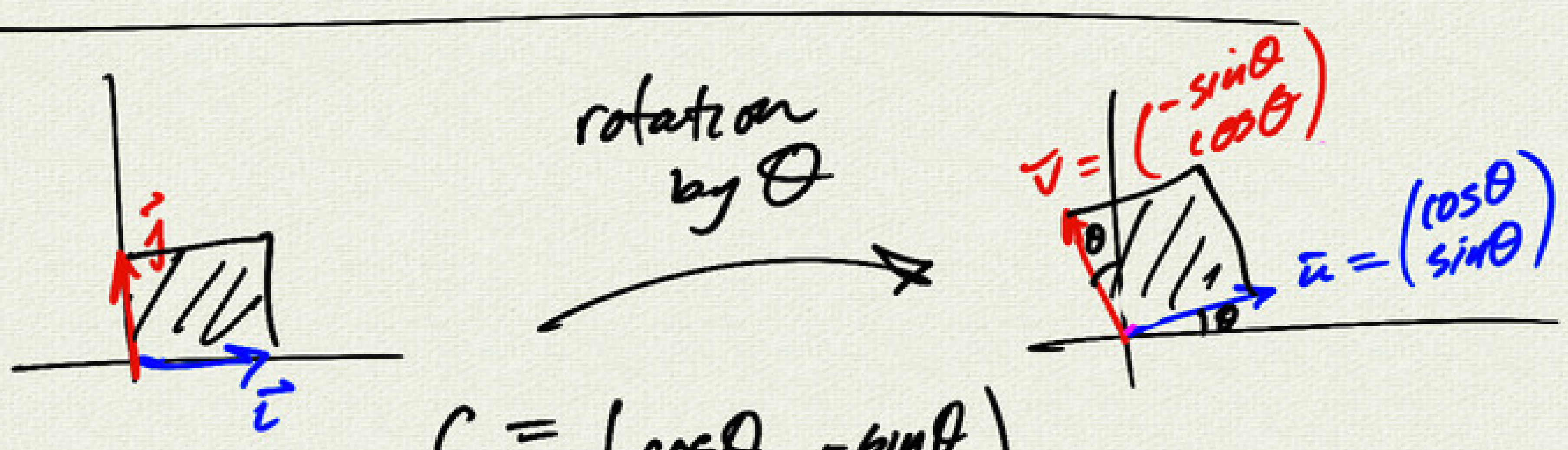
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \det A = 4$$

$$A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad \det A^{-1} = 1/4$$



$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \det B = -1$$

$B^{-1} = B$ check: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = I$ ✓



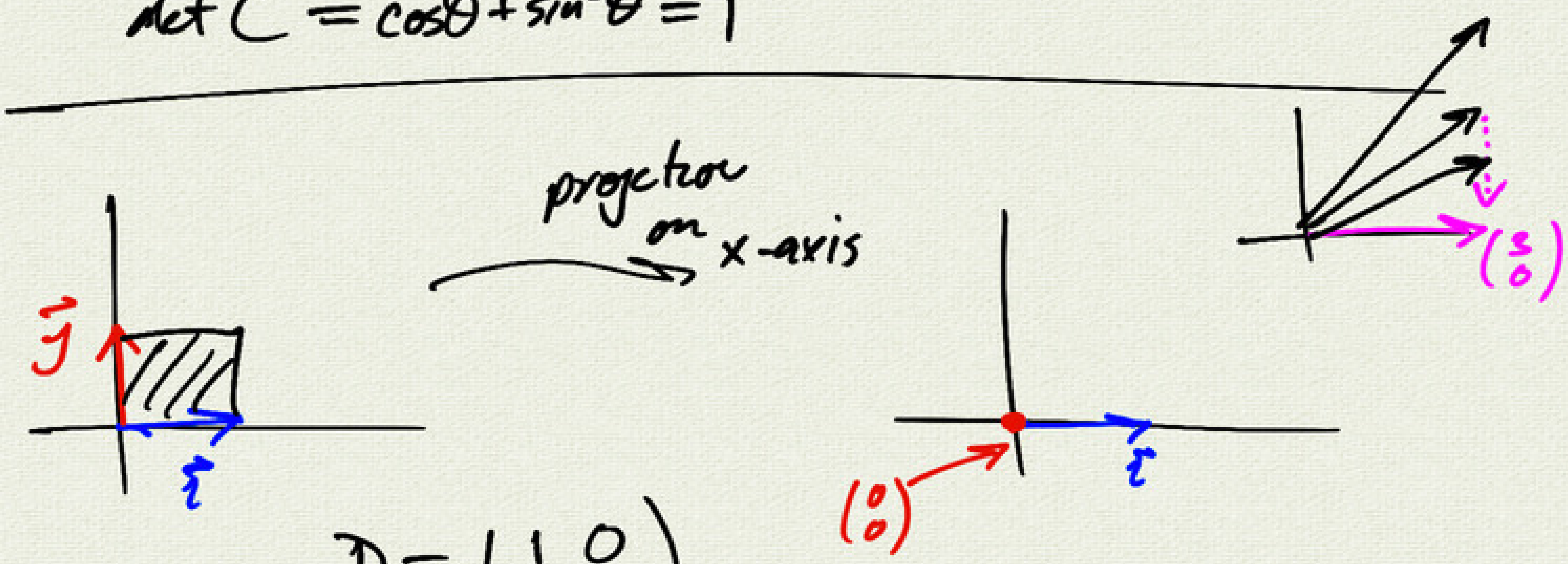
$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

C^{-1} : rotate by $-\theta$

$$C^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

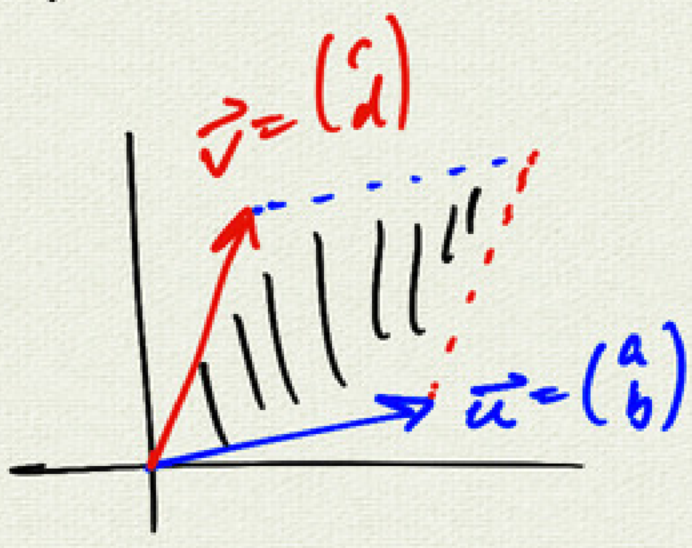
$$\det C = \cos^2\theta + \sin^2\theta = 1$$



$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\det D = 0 \implies D^{-1}$ does not exist

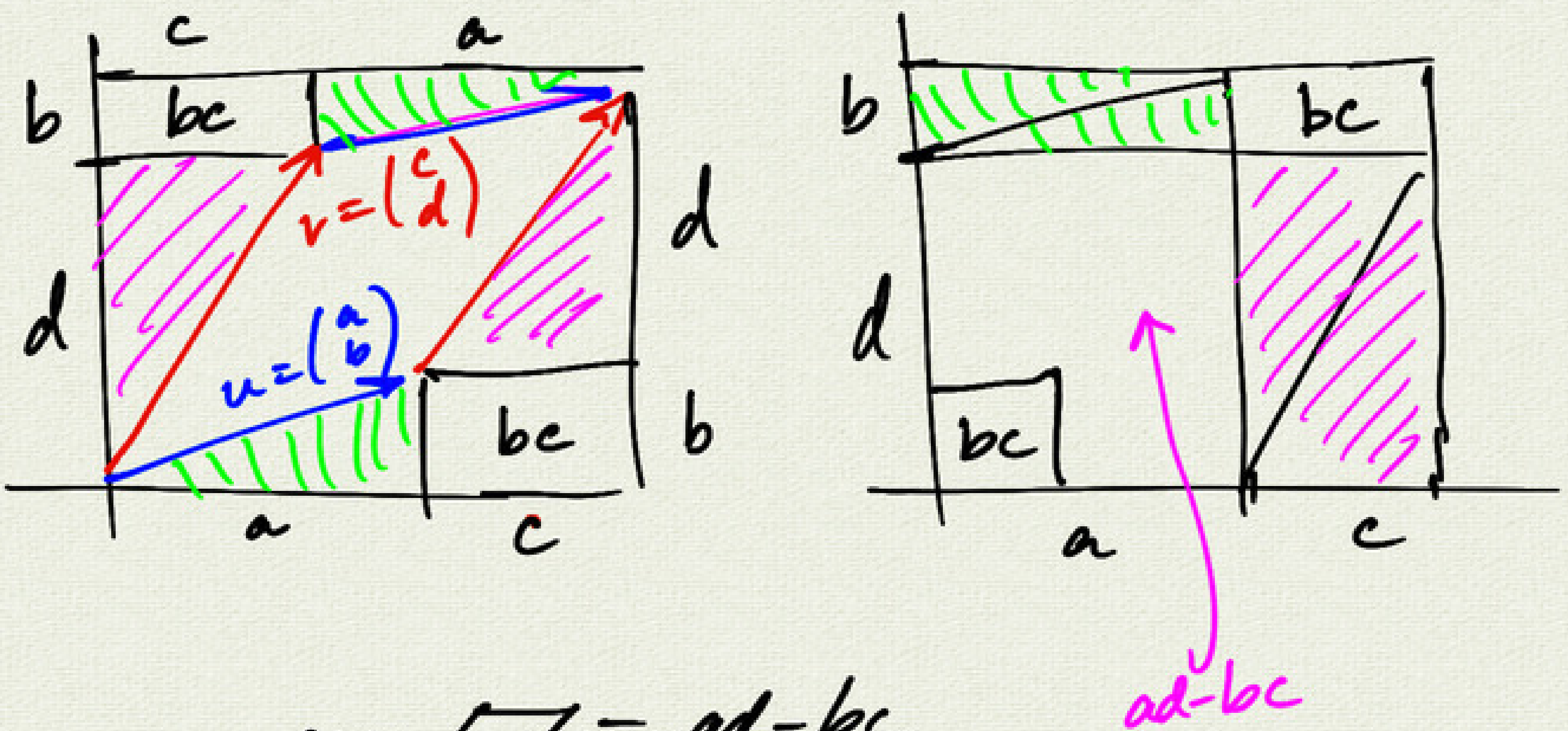
$\det A$ = area magnification



$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\det A$ = area ▱

(3D: volume of parallelopiped)



area ▱ = ad − bc