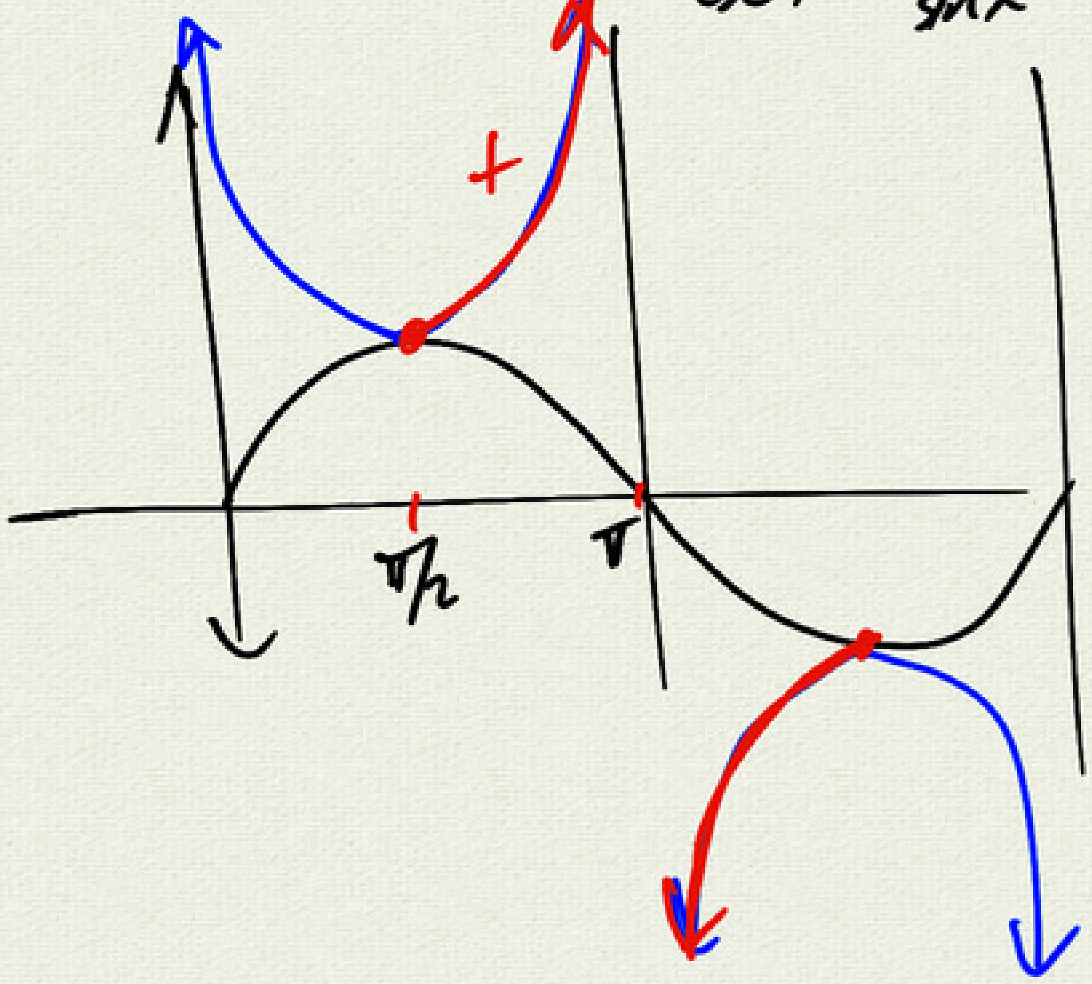
$\csc x = \frac{1}{\sin x}$



increasing on $[\pi/2, \pi)$

and on $(\pi, 3\pi/2]$

$(+2\pi k)$

$(\pi + 2\pi k, \frac{3\pi}{2} + 2\pi k]$

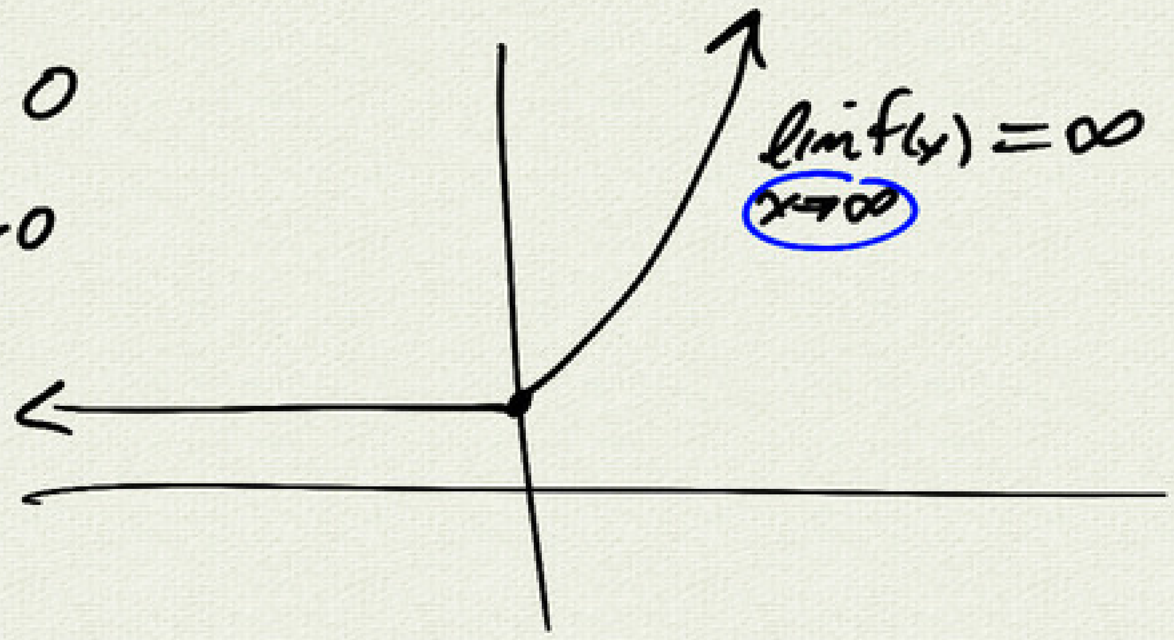
---

end behavior:

no limit

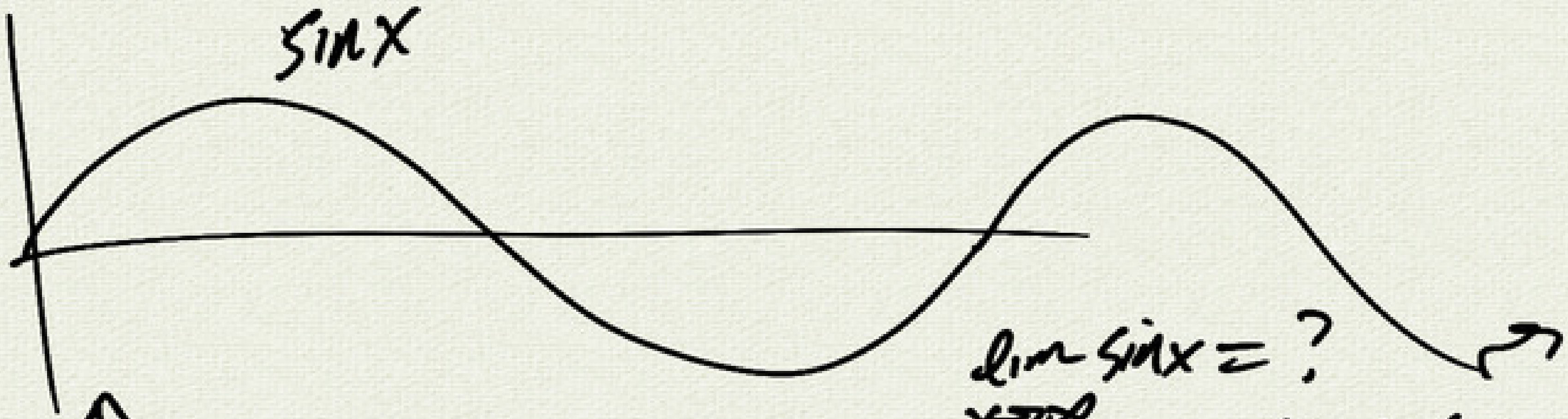
limit does not exist

$f(x) = \begin{cases} 1 & x \le 0 \\ 2^x & x > 0 \end{cases}$



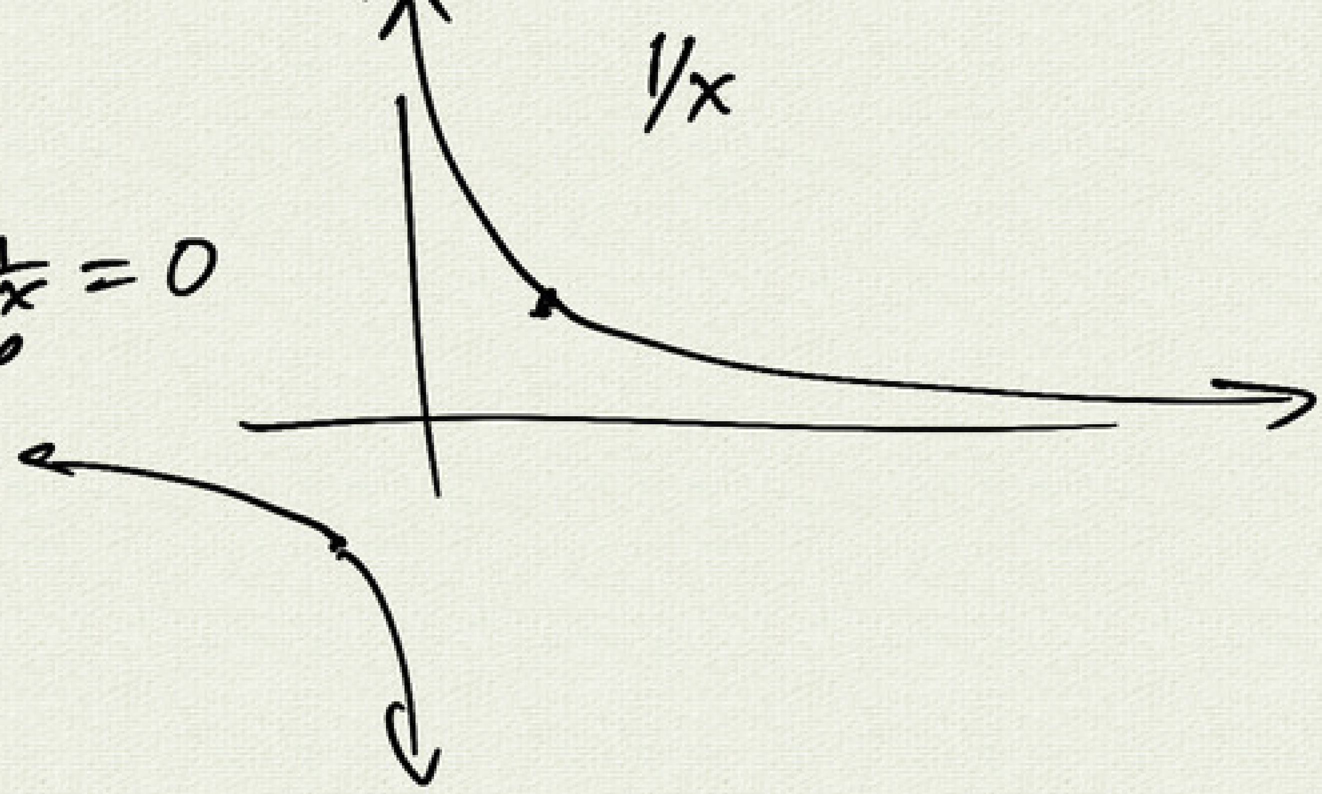
$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = 1$



$\lim_{x \to \infty} \sin x = ?$

limit does not exist

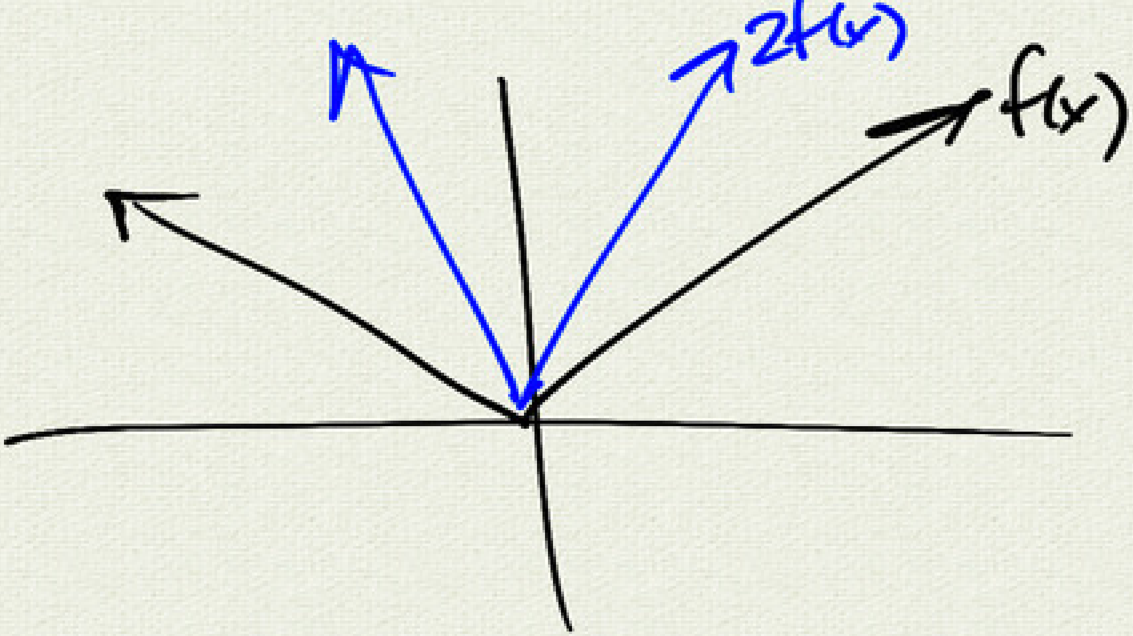


$\lim_{x \to -\infty} \frac{1}{x} = 0$

$\lim_{x \to \infty} \frac{1}{x} = 0$

(23) $f(x) = |x|$

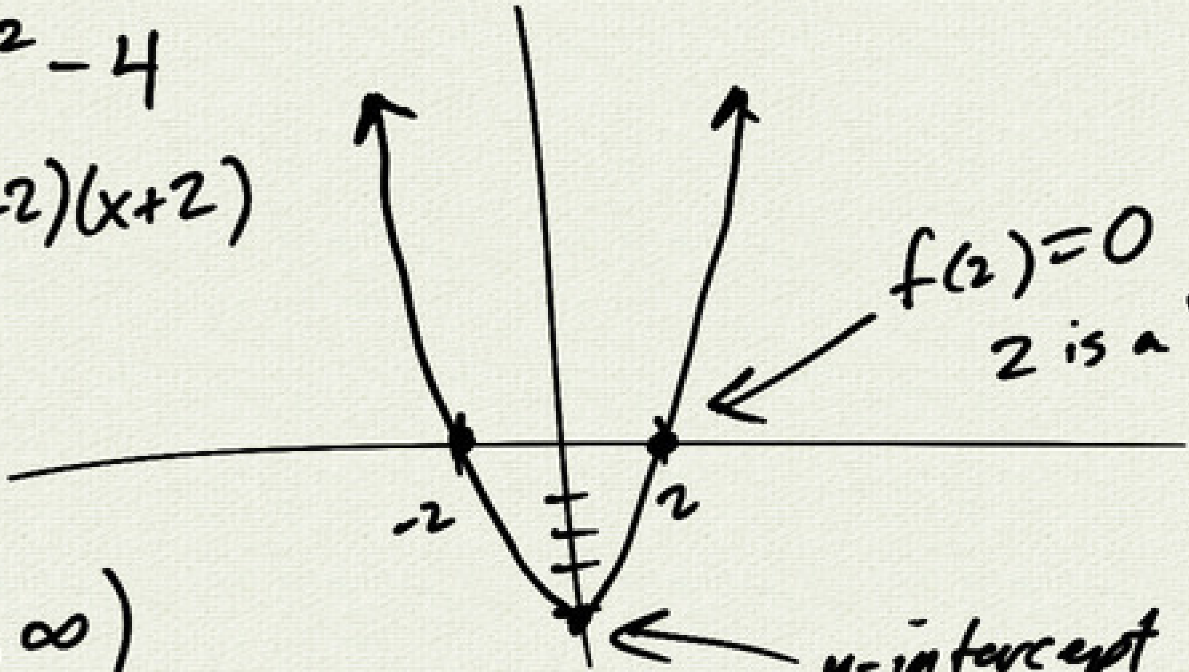
$2f(x)$



4.3 Polynomials

$f(x) = x^2 - 4$

$= (x-2)(x+2)$



$f(2) = 0$

2 is a "zero"

"root"

x-intercept

y-intercept

$f(0)$

domain $= \mathbb{R}$

range $= [-4, \infty)$

bounded below

increasing on $[0, \infty)$

decreasing $(-\infty, 0]$

global min at $x = 0$

no local max

end behavior: $\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

even symmetry:

$f(-x) = f(x)$

check: $f(-x) = (-x)^2 - 4$

$= x^2 - 4$

$= f(x)$ ✓

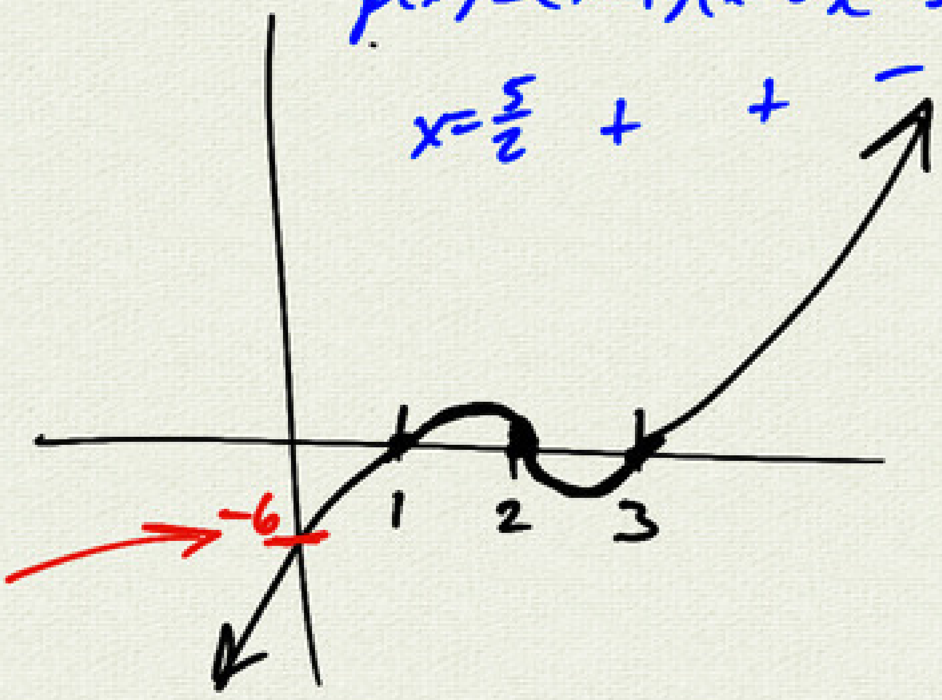
$x = 3/2 \quad + \quad - \quad -$

$p(x) = (x-1)(x-2)(x-3)$

$x = \frac{5}{2} \quad + \quad + \quad -$

another example:

$$p(x) = (x-1)(x-2)(x-3)$$
$$= (x-1)(x^2-5x+6)$$
$$= x^3-5x^2+6x$$
$$\quad -x^2+5x-6$$
$$= \boxed{x^3} - 6x^2 + 11x - 6$$



zeros: $x = 1, 2, 3$

leading term $\Rightarrow$ end behavior $\quad \lim_{x\to\infty} p(x) = \infty$

$\lim_{x\to-\infty} p(x) = -\infty$

constant term $\Rightarrow$ y-intercept

---

polynomial division:

$$\begin{array}{r|l} & \quad\; x - 1 \\ \hline x^2-5x+6 & x^3 - 6x^2 + 11x - 6 \\ & x^3 - 5x^2 + 6x \\ \hline & \quad -x^2 + 5x - 6 \\ & \quad -x^2 + 5x - 6 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

$0 \Leftarrow$ remainder

$$p(x) = \underbrace{(x-1)}_{\text{quotient}} \underbrace{(x^2-5x+6)}_{\text{divisor}} + \underbrace{0}_{\text{remainder}}$$

---

elementary school: $\quad 53 \div 5$

$$53 = \underbrace{10}_{\text{quotient}} \cdot \underbrace{5}_{\text{divisor}} + \underbrace{3}_{\text{remainder}}$$

$$p(x) = x^3 - 6x^2 + 11x - 6$$

$$d(x) = x^2 - 5x + 6$$

$$h(x) = p(x) + (x + 1)$$

$$= x^3 - 6x^2 + 12x - 5$$

$$h(x) \div d(x)$$

$$\begin{array}{r|l} & x - 1 \\ \hline \underbrace{x^2 - 5x + 6}_{\substack{\text{divisor} \\ d(x)}} & x^3 - 6x^2 + 12x - 5 \\ & \underline{x^3 - 5x^2 + 6x} \\ & -x^2 + 6x - 5 \\ & \underline{-x^2 + 5x - 6} \\ & \boxed{x + 1} \end{array}$$

quotient $q(x)$

remainder $r(x)$

result:

$$x^3 - 6x^2 + 12x - 5 = (x-1)(x^2 - 5x + 6) + (x+1)$$

$$p(x) = q(x) \quad d(x) + r(x)$$

or:

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

$$\frac{x^3 - 6x^2 + 12x - 5}{x^2 - 5x + 6} = \underbrace{x - 1}_{\text{quotient}} + \underbrace{\frac{x+1}{x^2 - 5x + 6}}_{\text{remainder as a fraction}}$$

degree 1

degree 2

$$53 = 10 \cdot 5 + 3$$

$$\frac{53}{5} = \underbrace{10}_{\text{quotient}} + \underbrace{\frac{3}{5}}_{\text{remainder as a fraction}}$$

notation:

polynomial

degree(p) = n

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

leading term → end behavior

leading coefficient

constant term ⇓ y-intercept

division: given p(x) polynomial
d(x) divisor (polynomial)

we can write

$$p(x) = q(x)\,d(x) + r(x)$$

polynomial, quotient, divisor, remainder

with $\deg(r(x)) < \deg(d(x))$

$p(x) = (x-1)(x-2)(x-3)$ factored $\rightarrow$ zeros at $x = 1, 2, 3$

$\deg(p) = 3 \Rightarrow$ at most 3 zeros

$\deg(p) = n \Rightarrow$ at most n zeros

$(x-a)$ factor of $p(x) \Rightarrow$ $x = a$ is a zero

$p(a) = 0$

Remainder Theorem:

polynomial $p(x)$

consider $x = a$

divide $p(x)$ by $x - a$:

for any a,

$p(a) = r$,

where r is the remainder from dividing by $x - a$

$$p(x) = q(x)\underbrace{(x-a)}_{\text{divisor}} + r(x)$$

$\deg(r) < 1$

$\deg(r) = 0$

r constant

$$p(x) = \underbrace{q(x)(x-a)} + r$$

$\Rightarrow p(a) = r$

---

example: $p(x) = (x-1)(x-2) + 3$

$= x^2 - 3x + 2 + 3$

$= x^2 - 3x + 5$

$a = 1$

$$\frac{p(x)}{x-1} = x - 2 + \frac{3}{x-1}$$ remainder

also: $p(1) = 3$

Factor Theorem:

suppose $p(x)$ polynomial and $p(a) = 0$

then $p(x) = q(x)(x-a) + r$

$p(a) = r = 0$

$\Rightarrow$ $x - a$ divides $p(x)$ evenly

$x - a$ is a factor

$x - a$ is a factor of $p(x) \Rightarrow p(a) = 0$

a is a zero / root

before

now: $p(a) = 0 \Leftrightarrow x - a$ is a factor

a is a zero

a is a root

$x - a \mid p(x)$

"$x-a$ divides $p(x)$"

synthetic division : division by $x-a$ | linear = deg 1

$$p(x) = (x-1)(x-2)(x-3)$$
$$= x^3 - 6x^2 + 11x - 6$$

divide by $x-3$:

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & 3 & -9 & 6 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

quotient remainder

$$p(x) = (x^2 - 3x + 2)(x-3)$$

---

divide by $x-1$:

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & \boxed{1} & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

---

divide by $x-2$:

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 11 & -6 \\ & & 2 & -8 & 6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$