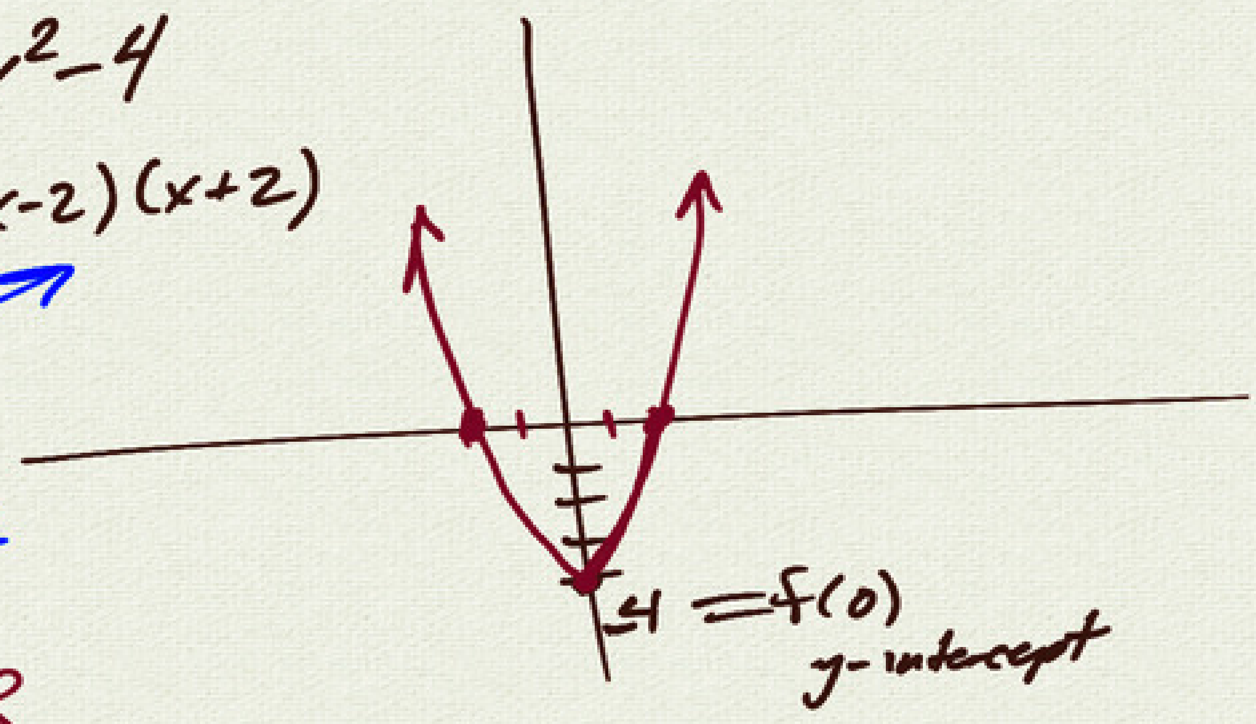
4.3 Polynomials

$f(x) = x^2 - 4$

$= (x-2)(x+2)$

zeros
roots
at $x = 2, -2$



domain $\mathbb{R}$

range $[-4, \infty)$

bounded below

no local or global max

global min at $x = 0$

end behavior:

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

$g(x) = (x-1)(x-2)(x-3)$

$= (x-1)(x^2 - 5x + 6)$

$= x^3 - 5x^2 + 6x$

$\quad - x^2 + 5x - 6$

$= x^3 - 6x^2 + 11x - 6$

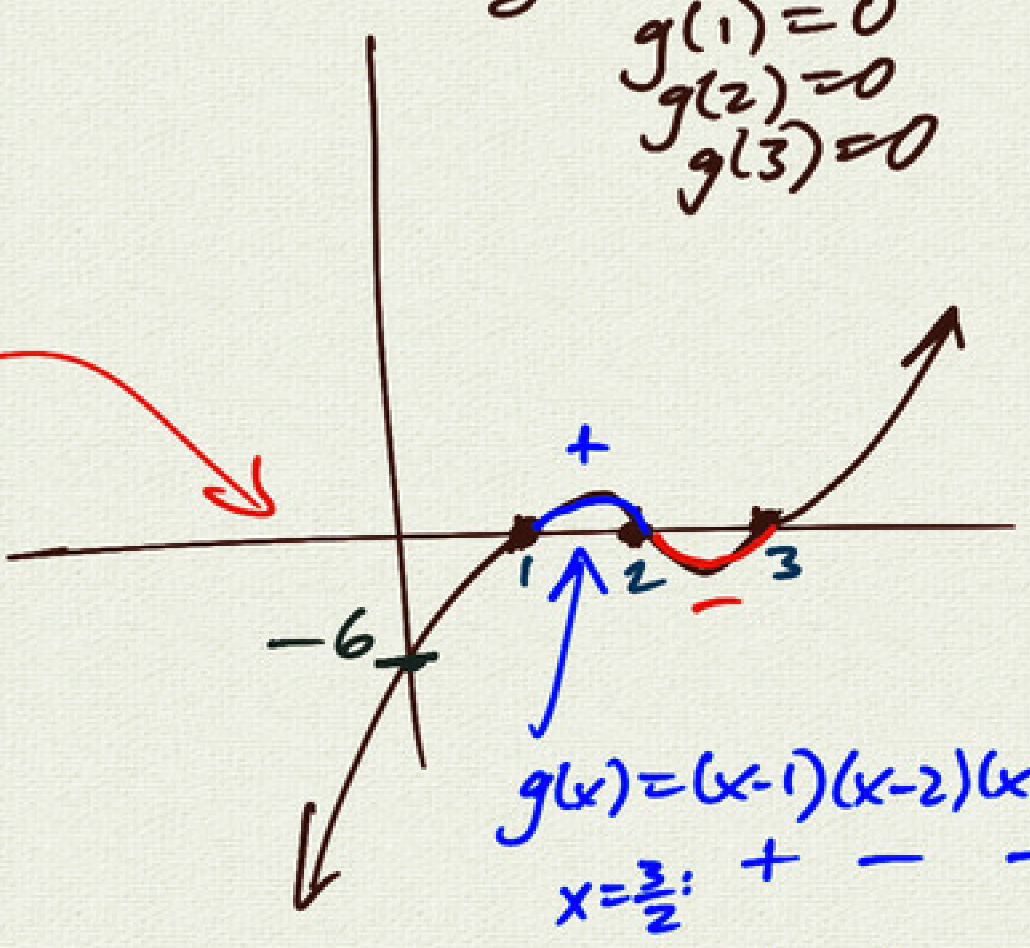
degree 3

zeros:

$g(1) = 0$

$g(2) = 0$

$g(3) = 0$



$g(x) = (x-1)(x-2)(x-3)$

$x = \frac{3}{2}$: $+ \quad - \quad -$

$x = \frac{5}{2}$: $+ \quad + \quad -$

division:

$$\begin{array}{r} x - 1 \quad \text{quotient} \\ \text{divisor } x^2 - 5x + 6 \overline{)\, x^3 - 6x^2 + 11x - 6} \\ \underline{x^3 - 5x^2 + 6x} \\ -x^2 + 5x - 6 \\ \underline{-x^2 + 5x - 6} \\ 0 \text{ remainder} \end{array}$$

$\Rightarrow x^3 - 6x^2 + 11x - 6 = \underbrace{(x^2 - 5x + 6)}_{\text{divisor}}\underbrace{(x-1)}_{\text{quotient}} + \underbrace{0}_{\text{remainder}}$

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elementary school: $53 \div 5$

$\longrightarrow \quad 53 = \underbrace{10}_{\text{quotient}} \cdot \underbrace{5}_{\text{divisor}} + 3 \longleftarrow$ remainder

remainder < divisor

$h(x) = g(x) + (x+1)$

$= (x^3 - 6x^2 + 11x - 6) + \boxed{(x+1)}$

$h(x) = x^3 - 6x^2 + 12x - 5$

$$\longrightarrow \quad \begin{array}{r} x - 1 \\ x^2 - 5x + 6 \overline{)\, x^3 - 6x^2 + 12x - 5} \\ \underline{x^3 - 5x^2 + 6x} \\ -x^2 + 6x - 5 \\ \underline{-x^2 + 5x - 6} \\ x + 1 \text{ remainder} \end{array}$$

result:

$x^3 - 6x^2 + 12x - 5 = \underbrace{(x-1)}_{\substack{\text{quotient} \\ q(x)}} \underbrace{(x^2 - 5x + 6)}_{\substack{\text{divisor} \\ d(x)}} + \underbrace{(x+1)}_{\substack{\text{remainder} \\ r(x)}}$

$\deg(r) < \deg(d)$

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for any polynomials $p(x)$, $d(x)$

we can write

$p(x) = q(x)d(x) + r(x)$ with $\deg(r) < \deg(d)$

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alternate way to express result:

$\dfrac{x^3 - 6x^2 + 12x - 5}{x^2 - 5x + 6} = \underbrace{x - 1}_{\text{quotient}} + \underbrace{\dfrac{x+1}{x^2 - 5x + 6}}_{\text{remainder as a fraction}}$

$53 = \underset{q}{10} \cdot \underset{d}{5} + \underset{r}{3}$

$\dfrac{53}{5} = \underset{q}{10} + \underbrace{\dfrac{3}{5}}_{\text{remainder as a fraction}}$

$$p(x) = (x-1)(x-2)(x-3)$$

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notation:

polynomial

$$p(x) = \underbrace{a_n x^{n}}_{\text{leading term}} + a_{n-1}x^{n-1} + \ldots + a_1 x + \underbrace{a_0}_{\text{constant term (y-intercept)}}$$

degree$(p) = n$

leading coefficient

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polynomial degree $n \Longrightarrow$

at most n zeros

$$p(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$$

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if $(x-a)$ is a factor of $p(x)$

then $p(a) = 0$

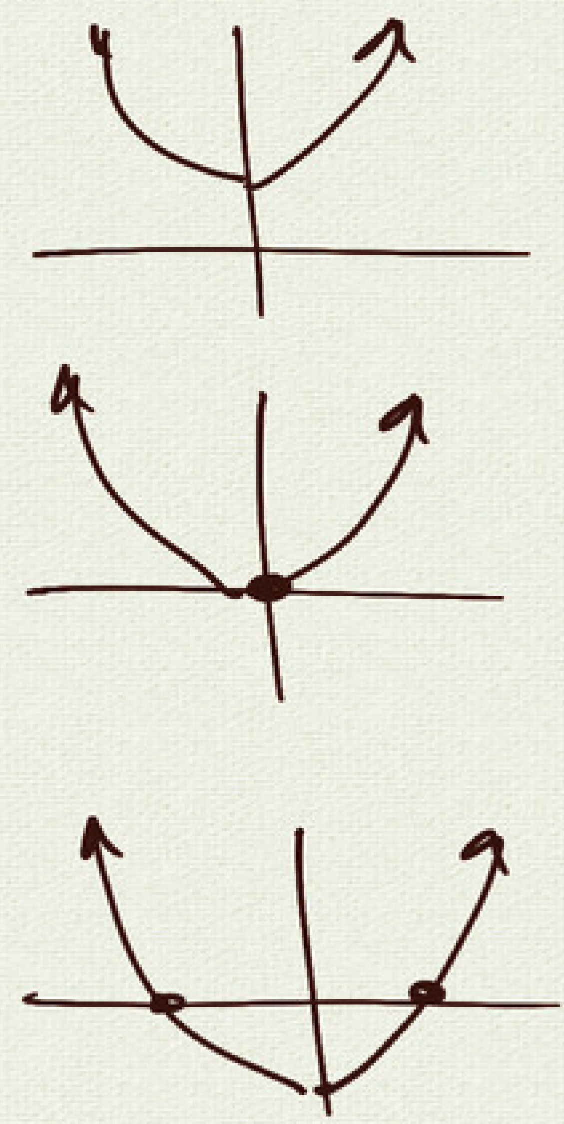
a is a zero

$p(x) = (x-a)q(x) \Longrightarrow p(a) = 0$

converse?

if a is a zero, then $x-a$ is a factor

degree 2:



Suppose $p(a) = 0$

divide $p(x)$ by $x-a$:

$p(x) = q(x)(x-a) + r(x)$

$\deg(r) < \deg(x-a) = 1$

$\deg(r) = 0$

r = constant (number)

$p(a) = r = 0$

$\Longrightarrow$ $\boxed{p(x) = q(x)(x-a)}$

$x-a$ is a factor of $p(x)$

Show: $x-a \mid p(x)$

"$x-a$ divides $p(x)$"

"$x-a$ is a factor"

remainder theorem

$p(a) = r$, where

r is the remainder when you divide $p(x)$ by $x-a$

Factor Theorem

$p(a) = 0 \Longleftrightarrow x-a \mid p(x)$

a is root / zero

$x-a$ is a factor

$$p(x) = x^3 - 6x^2 + 11x - 6 \quad \left(= (x-1)(x-2)(x-3) \right)$$

$\dfrac{p(x)}{(x-3)}$: synthetic division

zero

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & 3 & -9 & 6 \\ \hline & \underbrace{1 \quad -3 \quad 2}_{\text{quotient}} & & & \boxed{0} \end{array}$$

remainder

$x^2 - 3x + 2$

$$\Rightarrow p(x) = (x^2 - 3x + 2)(x - 3)$$

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practice:

$p(x) \div (x-2)$.

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 11 & -6 \\ & & 2 & -8 & 6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

$x^2 - 4x + 3$

quotient