4.5 Fundamental Theorem of Algebra

Factor Theorem

$p(a) = 0 \iff x-a \mid p(x)$

a is a zero / root

x−a is a factor

$p(x)$ polynomial
$\deg(p) = n$ $\Longrightarrow$ p has at most n real linear factors $(x-a)$ $\Longrightarrow$ p has at most n real roots

last time: $p(x) = \underbrace{(x-2)(x+2)(x-3)}_{\text{linear factors}}\underbrace{(x^2+1)}_{\text{irreducible quadratic}}$

3 real roots

$x^2+1=0$

$x^2=-1$

$x = \pm i \Rightarrow p(i)=0$, $p(-i)=0$

$p(x) = \underbrace{(x-2)(x+2)(x-3)(x-i)(x+i)}_{\text{5 complex linear factors}}$

$\rightarrow$ 5 complex roots

Fundamental Theorem of Algebra:

$p(x)$ polynomial
$\deg(p) = n$ $\Longrightarrow$ p has exactly n complex roots

n complex linear factors

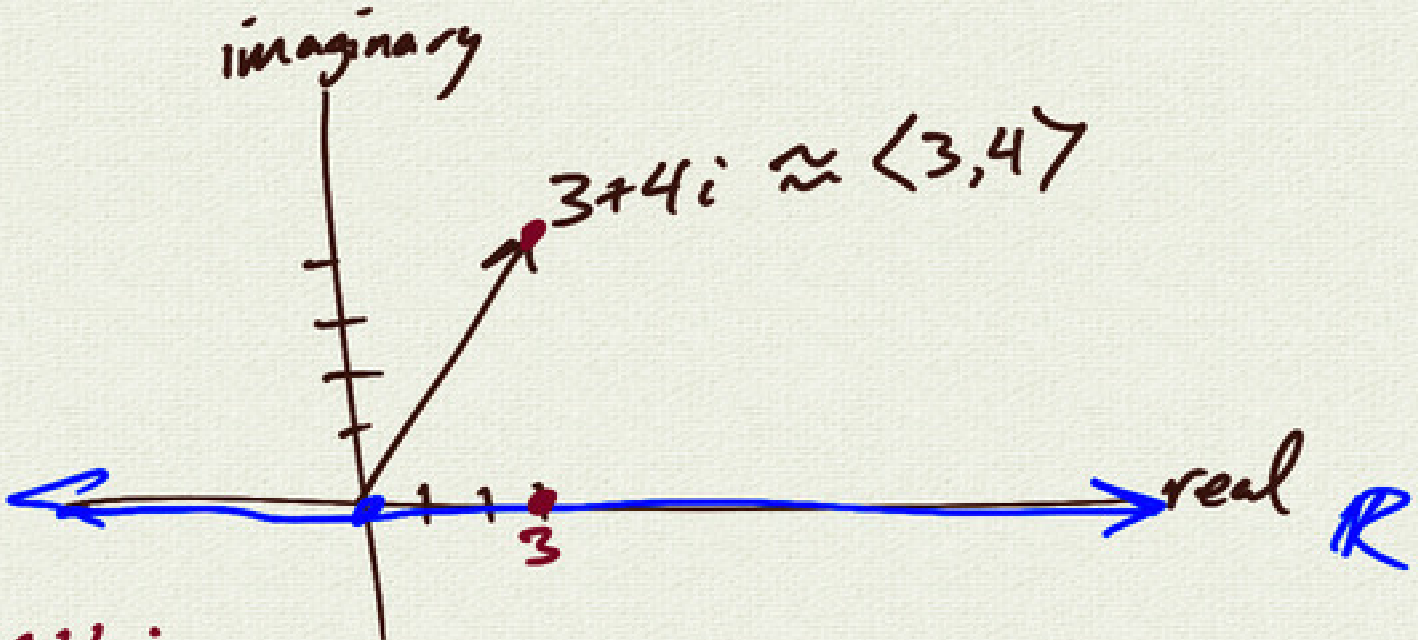
complex numbers

$z = a + bi$

real part — imaginary part

$$\left.\begin{aligned}(3+4i)+(5+6i) &= 8+10i \\ 2(3+4i) &= 6+8i\end{aligned}\right] \text{ just like vectors}$$

$a+bi \iff \langle a, b \rangle$



FOIL:

$(3+4i)(2+3i) = 6 + 9i + 8i + \underbrace{12i^2}_{-12}$

$= -6 + 17i$

$i^2 = -1$

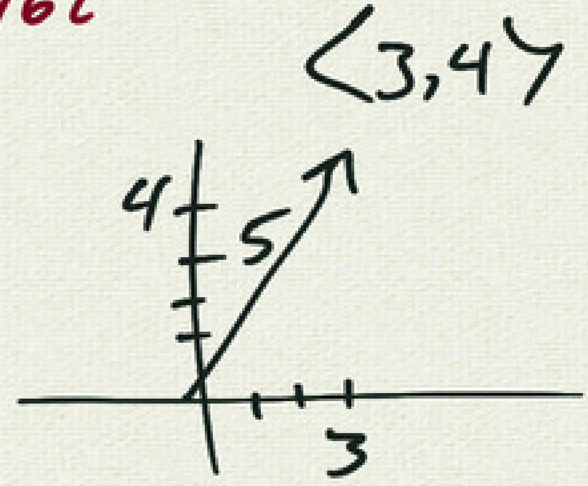
$i^3 = i^2 \cdot i = -i$

$i^4 = 1$

$(3+4i)\underbrace{(3-4i)}_{\text{complex conjugate}} = 9 - 12i + 12i - 16i^2$

$= 9 + 16$

$= 25$



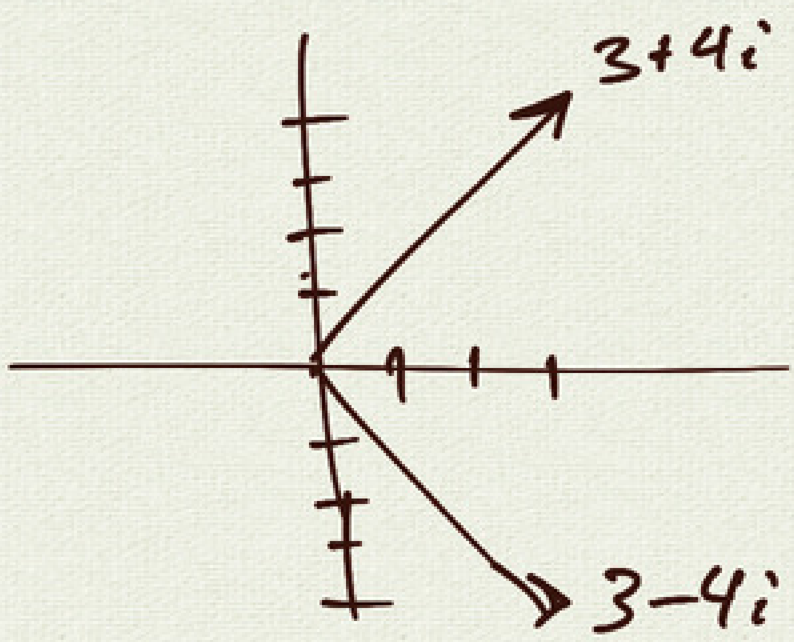
$z = a + bi$

$\Rightarrow \bar{z} = a - bi$ complex conjugate

$\boxed{z\bar{z} = |z|^2}$

$z = a + bi$ $(\approx \langle a, b \rangle)$

$\Rightarrow |z| = \sqrt{a^2 + b^2}$



example 1

$$p(x) = x^4 + 2x^2 + 8x + 5$$

factor completely (over $\mathbb{R}$ and $\mathbb{C}$)
real #'s   complex #'s

find all roots

potential rational roots: $\pm 1, 5$

$p(1) = 1+2+8+5 \neq 0$

$p(-1) = 1+2-8+5 = 0$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & 2 & 8 & 5 \\ & & -1 & 1 & -3 & -5 \\ \hline & 1 & -1 & 3 & 5 & \boxed{0} \end{array}$$

$p_2(x) = x^3 - x^2 + 3x + 5$ $p(x) = (x+1)p_2(x)$

$p_2(1) \neq 0$

$p_2(-1) = -1-1-3+5 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 3 & 5 \\ & & -1 & 2 & -5 \\ \hline & 1 & -2 & 5 & \boxed{0} \end{array}$$

$x^2 - 2x + 5 \implies p(x) = (x+1)^2(x^2-2x+5)$

$x^2 - 2x + 5 = 0$

$$\implies x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-20}}{2}$$

$$= 1 \pm \sqrt{-4}$$

$$= 1 \pm 2i$$ complex conjugates

$$\frac{\sqrt{-16}}{2} = \sqrt{-4}$$

$$= \sqrt{\frac{-16}{4}}$$

or

$$\pm \frac{4i}{2}$$

$$p(x) = (x+1)^2(x-(1+2i))(x-(1-2i))$$

complete factorization over $\mathbb{C}$

roots: $-1, -1, 1+2i, 1-2i$

Fundamental Theorem: $\deg(p) = n$

$\rightarrow$ p has exactly n complex roots

$p(x)$ has real coefficients

and non-real roots occur in complex conjugate pairs example 2 $\quad q(x) = x^4 - 6x^3 + x^2 + 54x - 90$

hint: $3+i$ is a root $\quad (\Rightarrow 3-i$ is also a root$)$

$$
\begin{array}{r|rrrrr}
3+i & 1 & -6 & +1 & 54 & -90 \\
 & & 3+i & -10 & -27-9i & 90 \\
\hline
3-i & 1 & -3+i & -9 & 27-9i & \boxed{0} \\
 & & 3-i & 0 & -27+9i & \\
\hline
 & 1 & 0 & -9 & \boxed{0} &
\end{array}
$$

$(3+i)(-3+i)$

$= -9 + \underbrace{i^2}_{-1}$

$= -10$

$(3+i)(27-9i)$

$= (3+i)(3-i)9$

$= (9-i^2)9$

$= 90$

$x^2 - 9 = (x-3)(x+3)$

$\Rightarrow p(x) = (x-3)(x+3)\boxed{(x-(3+i))(x-(3-i))}$

complete factorization over $\mathbb{C}$

roots: $3, -3, \underbrace{3+i, 3-i}_{\text{conjugate pair}}$

over $\mathbb{R}$: $(x-(3+i))(x-(3-i))$

$= x^2 - (3-i)x - (3+i)x + \underbrace{(3+i)(3-i)}_{9-i^2=10}$

$= x^2 - 6x + 10$

$\Rightarrow p(x) = (x-3)(x+3)(x^2-6x+10)$

complete factorization over $\mathbb{R}$

2 real roots: $3, -3$