4.6 Rational Functions

rational function

$$f(x) = \frac{p(x)}{q(x)} \leftarrow p, q \text{ polynomials}$$

rational number

- decimal expansion that ends or repeats

write as $\frac{a}{b}$ (fraction) $\leftarrow$ with $a, b \in \mathbb{Z}$

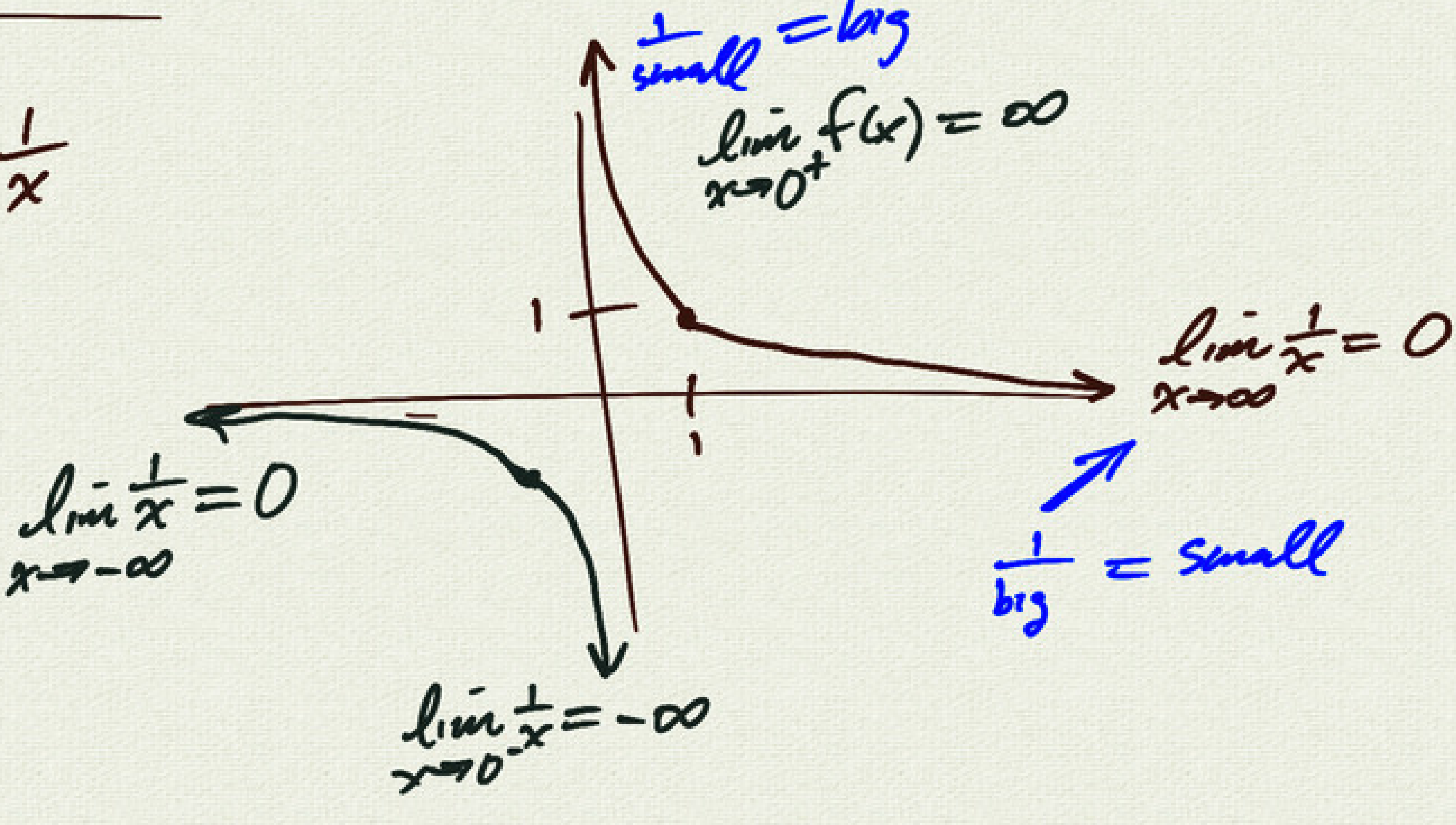
examples:

$f(x) = 1$

$g(x) = x^2$

$p(x)$ any polynomial

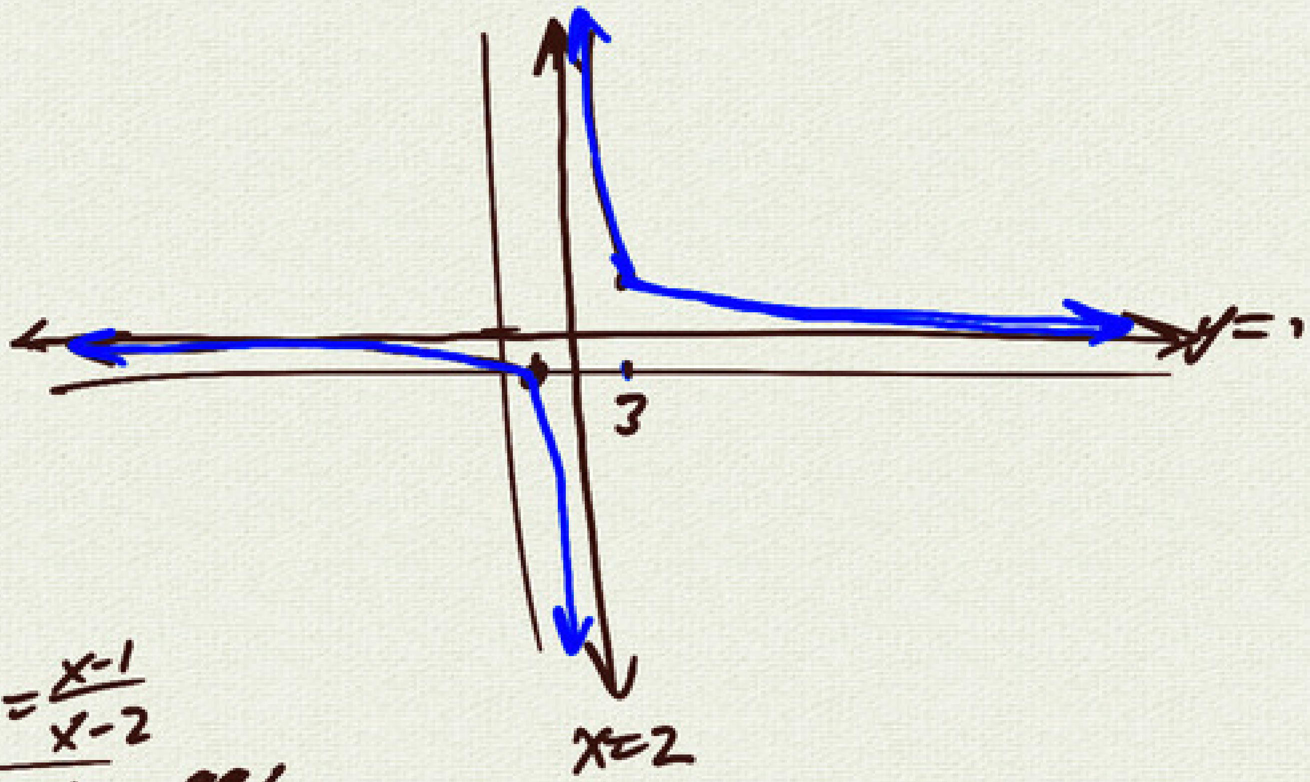
$$f(x) = \frac{1}{x}$$



example:

$$g(x) = \frac{x-1}{x-2}$$

$x \neq 2$ domain



end behavior:

$x \to \infty$

x	$g(x) = \frac{x-1}{x-2}$
100	$\frac{100-1}{100-2} = 99/98$
1000	$\frac{1000-1}{1000-2}$
1000 000	$\frac{1000000-1}{1000000-2} \approx 1$

$$\lim_{x \to \infty} g(x) = 1$$

$$\lim_{x \to -\infty} g(x) = 1$$

sign table:

		1		2	
$x-1$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$g(x)$	$+$	0	$-$	X	$+$

$$g(x) = \frac{x-1}{x-2}$$

divide:

$$\begin{array}{r|rr} 2 & 1 & -1 \\ & & 2 \\ \hline & 1 & \boxed{1} \end{array}$$

$$\longrightarrow \boxed{g(x) = 1 + \frac{1}{x-2}}$$

looks like $\frac{1}{x}$

up 1, right 2

check: $1 + \frac{1}{x-2}$

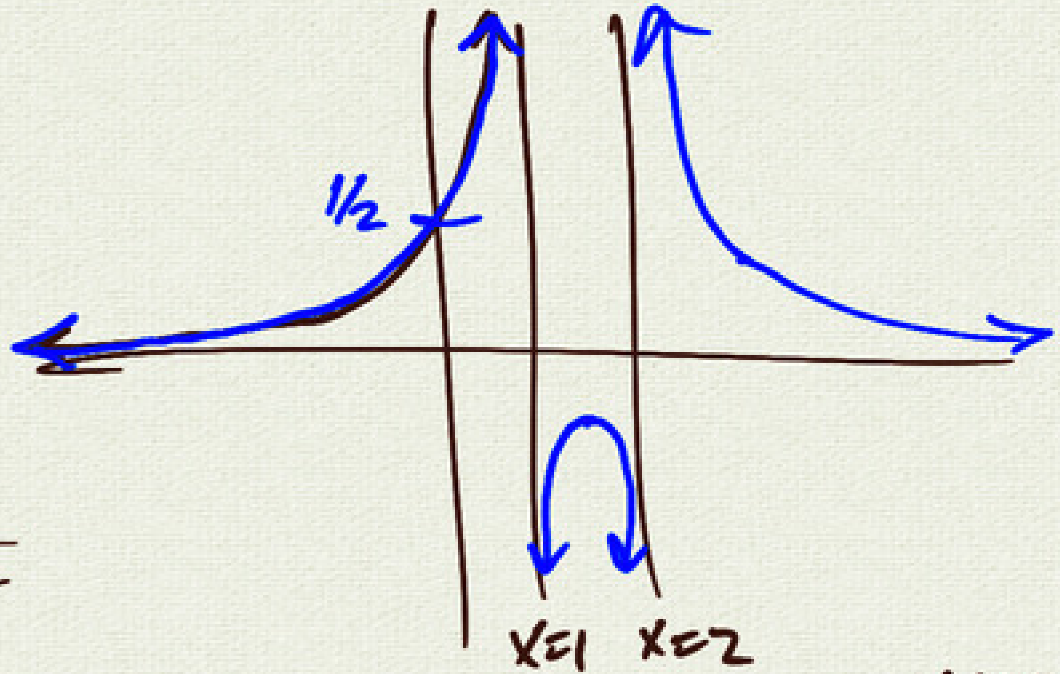
$$= \frac{x-2}{x-2} + \frac{1}{x-2}$$

$$= \frac{x-1}{x-2} \checkmark$$

example:

$$h(x) = \frac{1}{(x-1)(x-2)}$$

$$= \frac{1}{x^2-3x+2}$$



		1		2	
$x-1$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$h(x)$	$+$	X	$-$	X	$+$

end behavior:

$$\lim_{x \to \pm\infty} h(x) = 0$$

near $x=2$:

$$\lim_{x \to 2^+} h(x) = +\infty$$

$$\lim_{x \to 2^-} h(x) = -\infty$$

example:

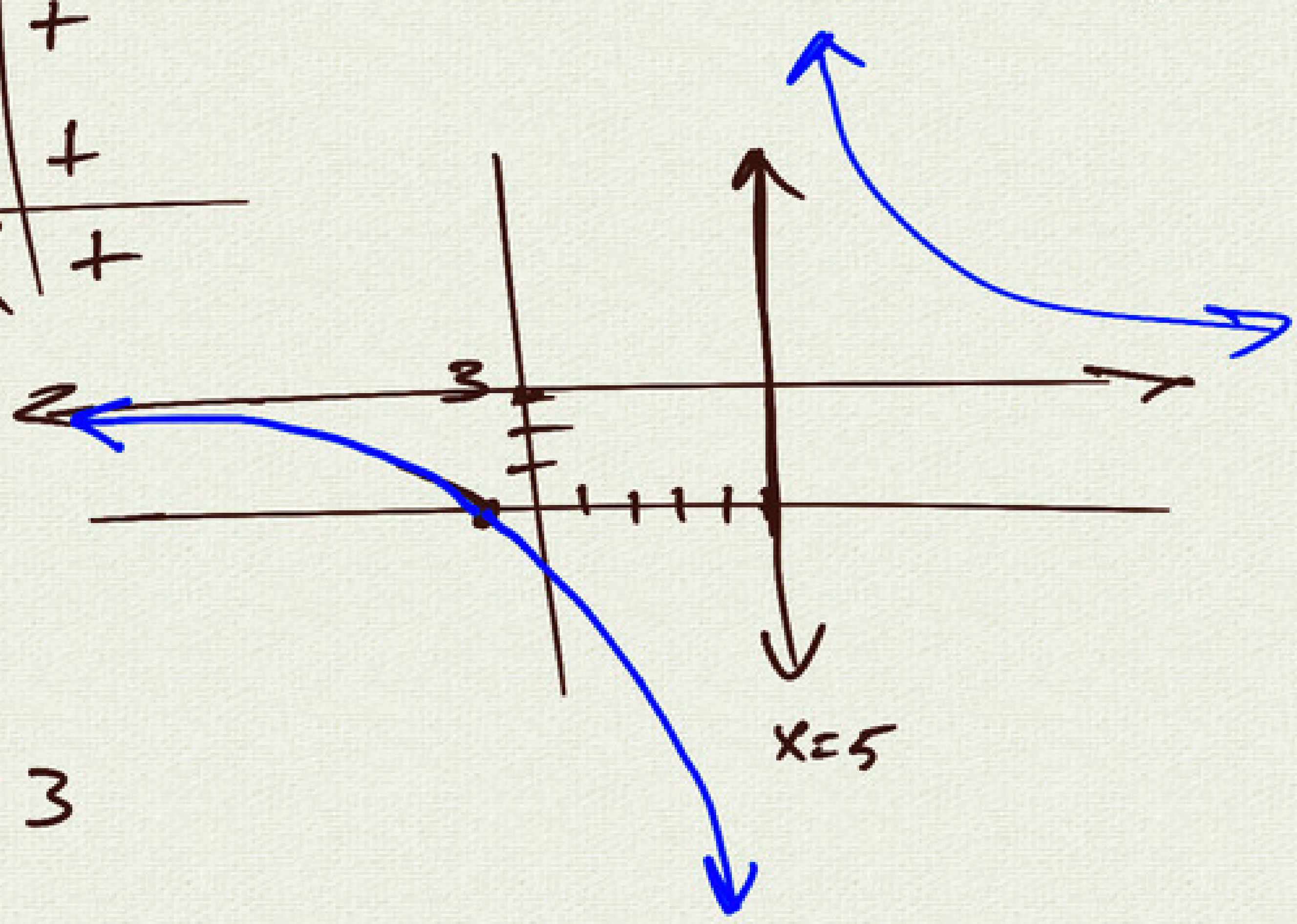
$$\boxed{f(x) = \frac{3x^2 - 3x - 6}{x^2 - 7x + 10}} \xrightarrow{\text{as } x \to \pm\infty} 3$$

$$= \frac{3(x^2 - x - 2)}{(x-5)(x-2)}$$

$$= \frac{3\cancel{(x-2)}(x+1)}{(x-5)\cancel{(x-2)}} = \begin{cases} \boxed{\dfrac{3(x+1)}{x-5}} & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$$

removable discontinuity

		-1		5	
$x+1$	−	0	+	+	+
$x-5$	−	−	−	0	+
$f(x)$	+	0	−	X	+



end behavior:

$$\frac{3(x+1)}{(x-5)} \longrightarrow ?$$

x	$\frac{3(x+1)}{x-5}$
1000000	$\frac{3(1000000+1)}{1000000-5} \approx 3$

$$\lim_{x \to \infty} f(x) = 3$$

$$\lim_{x \to -\infty} f(x) = -3$$

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rule for end behavior:

$$f(x) = \frac{p(x)}{q(x)} = \frac{\boxed{a_m x^m} + \ldots}{b_n x^n + \ldots}$$

$$\lim_{x \to \infty} f(x) = \begin{cases} (\pm)\infty & \text{if } m > n \\ a_m / b_n & m = n \\ 0 & m < n \end{cases}$$

$$g(x) = \frac{2x^2+2x-12}{x-1}$$

end behavior:

$$\lim_{x\to\infty} g(x) = \infty$$

$$\lim_{x\to-\infty} g(x) = -\infty$$

$$g(x) = \frac{2(x^2+x-6)}{x-1}$$

$$= \frac{2(x+3)(x-2)}{(x-1)}$$

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divide by $x-1$:

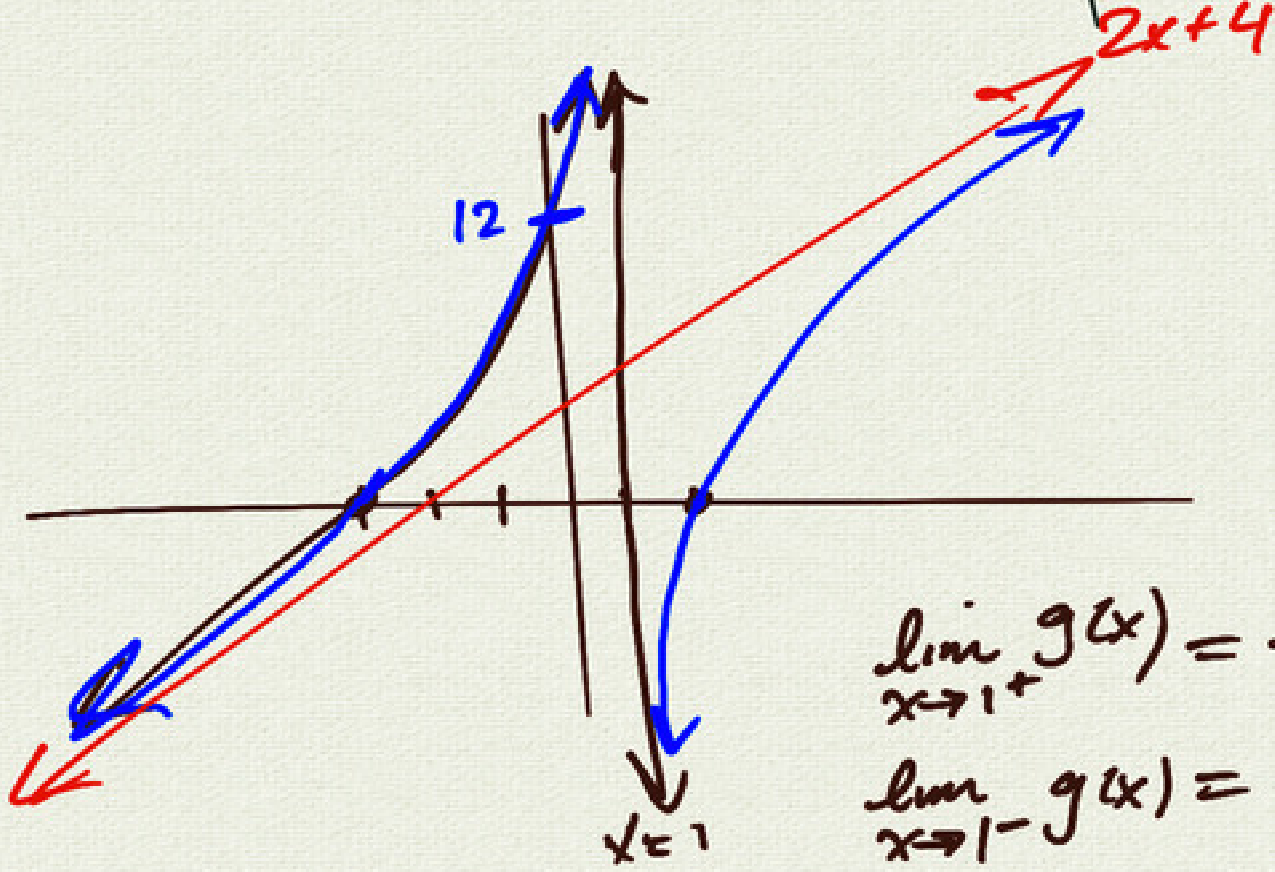
$$\begin{array}{r|rrr} 1 & 2 & 2 & -12 \\ & & 2 & 4 \\ \hline & 2 & 4 & \boxed{-8} \end{array}$$

$$\Rightarrow g(x) = 2x+4 \boxed{-\frac{8}{x-1}} \to 0 \text{ as } x\to\infty$$

$$\Rightarrow g(x) \approx 2x+4 \text{ as } x\to\infty$$

slant asymptote

		-3		1		2	
$x+3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$g(x)$	$-$	0	$+$	$\boxed{X}$	$-$	0	$+$



$$\lim_{x\to1^+} g(x) = -\infty$$

$$\lim_{x\to1^-} g(x) = +\infty$$