4.6 Rational Functions

rational number

$5 \leftarrow$ integers

(decimal expansion: terminates $\frac{1}{4} = .25$
or
repeats $\frac{1}{3} = .\overline{333}$)

fraction: $\frac{a}{b}$ $a, b \in \mathbb{Z}$

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rational function:

$$f(x) = \frac{p(x)}{q(x)} \qquad p, q \text{ polynomials}$$

Examples: $\frac{1}{x}$, any polynomial

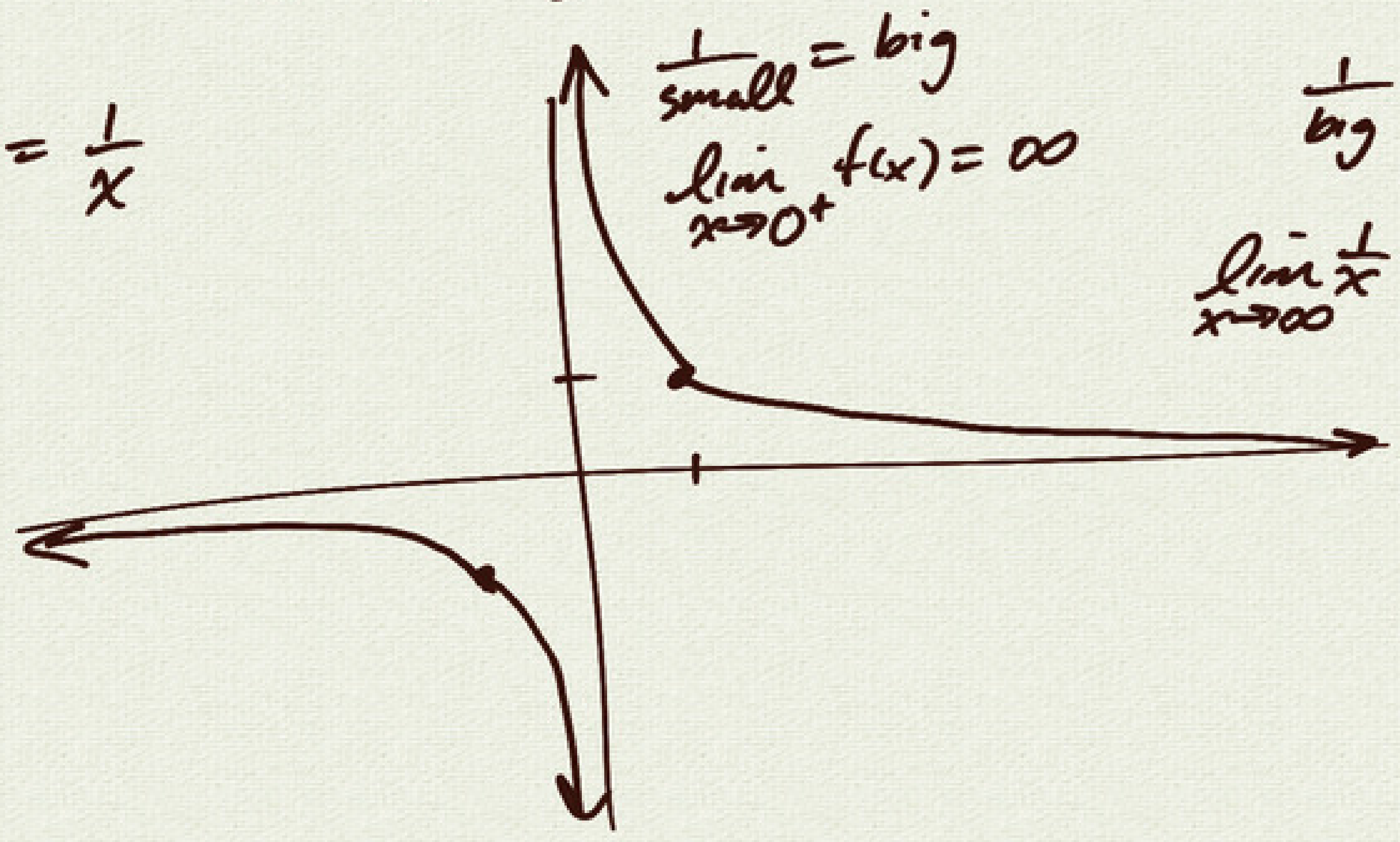
$f(x) = \frac{1}{x}$

$\frac{1}{\text{small}} = \text{big}$

$\lim_{x \to 0^+} f(x) = \infty$

$\frac{1}{\text{big}} = \text{small}$

$\lim_{x \to \infty} \frac{1}{x} = 0$

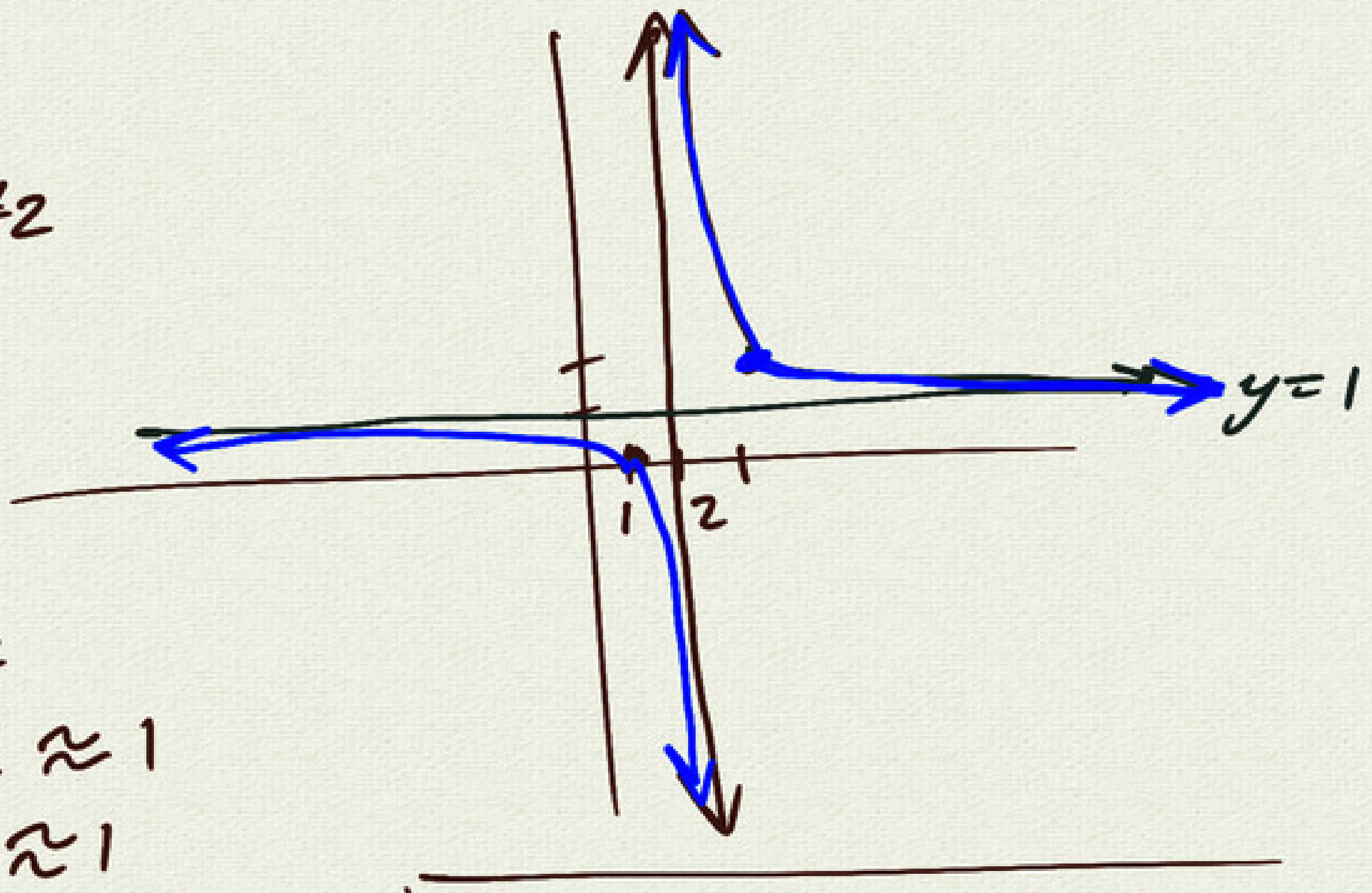


$g(x) = \frac{x-1}{x-2}$

domain: $x \neq 2$

$g(1) = 0$

$g(3) = 2$



x	$g(x) = \frac{x-1}{x-2}$	
1 000 000	$\frac{1000000-1}{1000000-2}$	≈ 1
1 billion		≈ 1

$\lim_{x \to \infty} \frac{x-1}{x-2} = 1$

$\lim_{x \to \infty} \frac{2x-1}{x-2} = 2$

		1		2	
$x-1$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$g(x)$	$+$	0	$-$	X	$+$

$g(x) = \frac{x-1}{x-2}$

division:
$$\begin{array}{r|rr} 2 & 1 & -1 \\ & & 2 \\ \hline & 1 & \boxed{1} \end{array}$$

$$\boxed{g(x) = 1 + \frac{1}{x-2}}$$

"$\frac{1}{x}$" shifted right 2 up 1

check:

$1 + \frac{1}{x-2} = \frac{x-2}{x-2} + \frac{1}{x-2}$

$= \frac{x-1}{x-2}$ ✓

example:

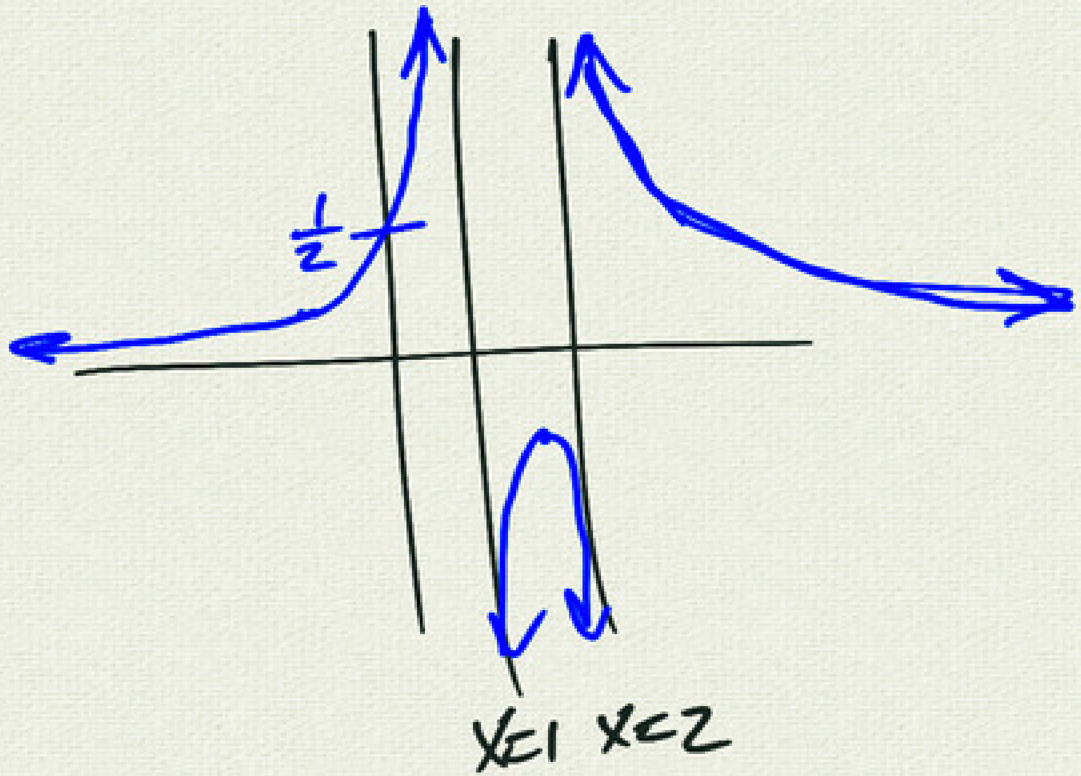
$$f(x) = \frac{1}{(x-1)(x-2)}$$

domain: $x \neq 1, 2$

end behavior:

$$\lim_{x \to \infty} f(x) = 0$$

		1		2	
$x-1$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	X	$-$	X	$+$



$$g(x) = \frac{3x^2-3x-6}{x^2-7x+10}$$

$$= \frac{3(x^2-x-2)}{(x-5)(x-2)}$$

$$= \frac{3(x-2)(x+1)}{(x-5)(x-2)} = \begin{cases} \boxed{\dfrac{3(x+1)}{x-5}} & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$$

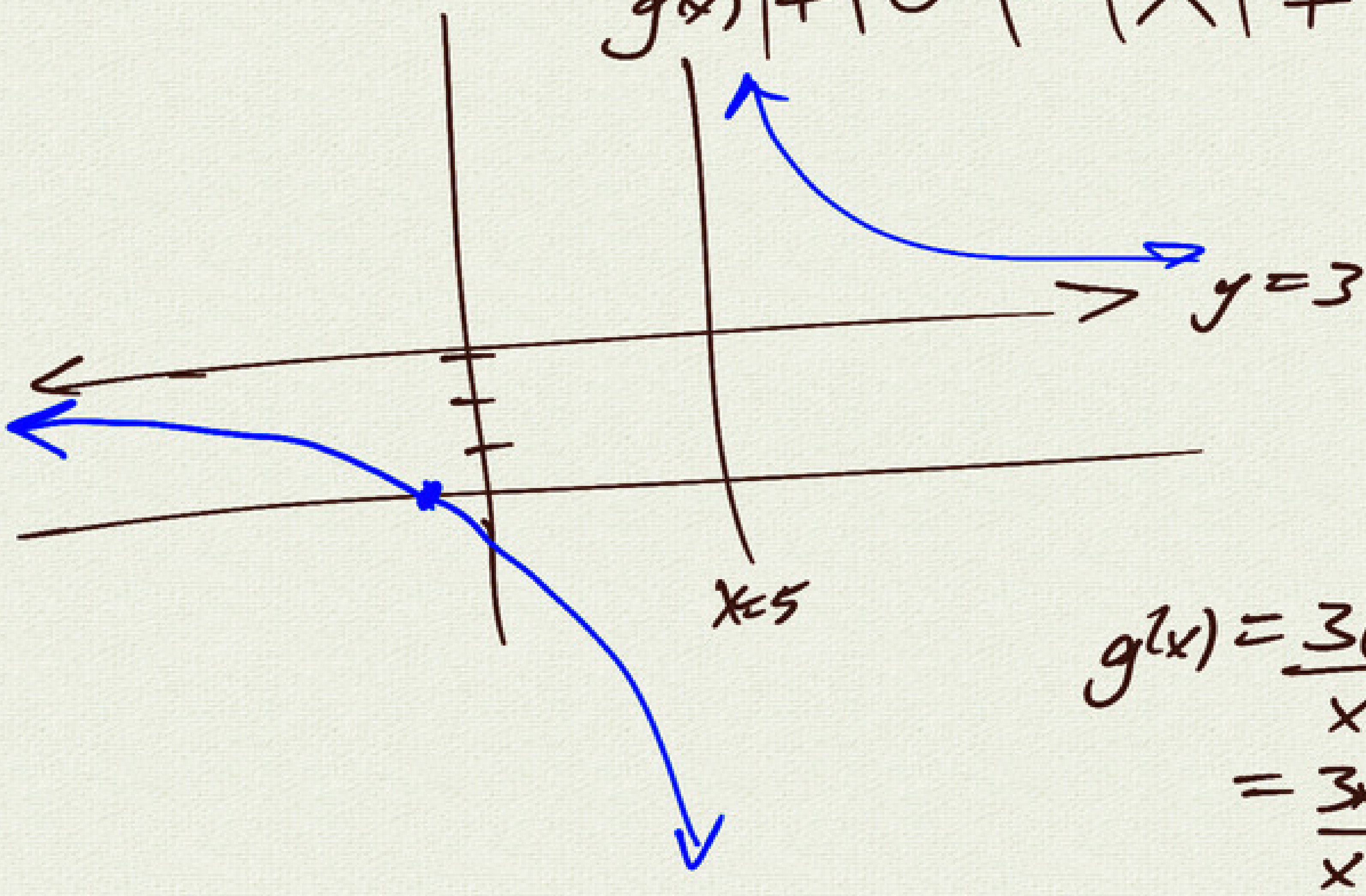
(removable discontinuity)

end behavior:

$$\lim_{x\to\infty} g(x) = 3$$

$$\lim_{x\to-\infty} g(x) = 3$$

		-1		5	
x+1	−	0	+	+	+
x−5	−	−	−	0	+
g(x)	+	0	−	X	+



$$g(x) = \frac{3(x+1)}{x-5}$$

$$= \frac{3x+3}{x-5}$$

x	g(x)
1000 000	$\frac{3,000,00+3}{1,000,000-5} \longrightarrow 3$

$$h(x) = \frac{2x^2 + 2x - 12}{x - 1}$$

end behavior:

x	$h(x)$
1000000	$\dfrac{2\boxed{(1000000)^2} + 2000000 - 12}{1000000}$

top increasing faster than bottom

$$\lim_{x \to \infty} h(x) = \infty$$

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rule: $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{ax^m + \dots}{bx^n + \dots}$

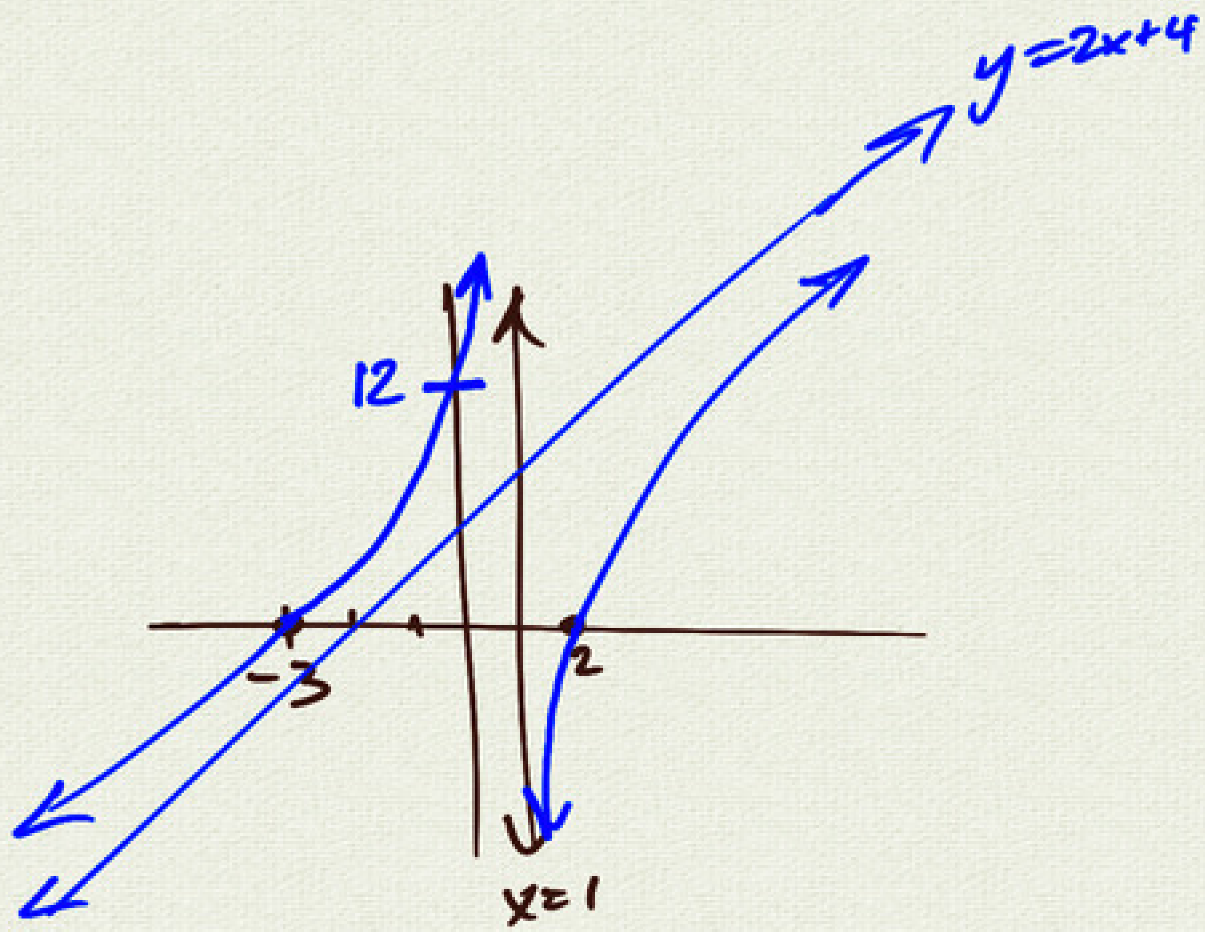
end behavior:

$$\lim_{x \to \infty} f(x) = \begin{cases} \infty & \text{if } m > n \\ \dfrac{a}{b} & \text{if } m = n \\ 0 & \text{if } m < n \end{cases}$$

$$h(x) = \frac{2x^2 + 2x - 12}{x-1}$$

$$= \frac{2(x^2 + x - 6)}{x-1}$$

$$= \frac{2(x+3)(x-2)}{x-1}$$



end behavior:

$$\lim_{x \to \infty} h(x) = \infty$$

$$\lim_{x \to -\infty} h(x) = -\infty$$

		-3		1		2	
x+3	−	0	+	+	+	+	+
x−1	−	−	−	0	+	+	+
x−2	−	−	−	−	−	0	+
h(x)	−	0	+	X	−	0	+

divide: $h(x) = \dfrac{2x^2 + 2x - 12}{x-1}$

$$\begin{array}{r|rrr} 1 & 2 & 2 & -12 \\ & & 2 & 4 \\ \hline & 2 & 4 & \boxed{-8} \end{array}$$

$2x+4$
quotient

$$\Longrightarrow h(x) = \underbrace{2x+4} + \boxed{\left(\frac{-8}{x-1}\right)} \to 0$$

as $x \to \infty$

$h(x) \approx 2x + 4$
as $x \to \infty$
slant asymptote