6.1 Combinatorics

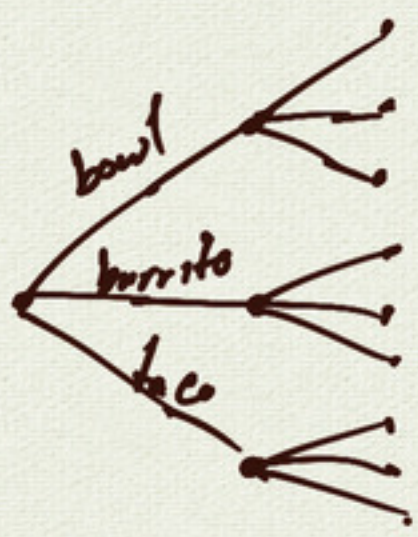
counting | discrete vs. continuous

example: Chipotle

(bowl, burrito, taco) (white rice, brown rice, none) (beef, chicken, tofu) (black beans, pinto beans, none) (cheese, no cheese) (lettuce, no)

how many different meals?

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2$



...

independent choices
$\Rightarrow$ multiply

(visualize as branching)

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pizza: 10 possible toppings

how many different pizzas?

$\underline{0/1} \; \underline{0/1} \; \underline{0/1} \; \underline{\quad} \; \underline{\quad} \; \cdots \; \underline{\quad} \; \underline{\quad} \; \underline{\quad} \; \underline{\quad} \; \underline{0/1}$

10 toppings

2^{10} possible pizzas

$2^{10} = 1024 \approx 1000$

binary sequence $\longleftrightarrow$ pizza

00000 00000	plain
11111 11111	supreme
11000 00000	cheese + tomato

$$\#\text{binary sequences length 10} = 2^{10} = \#\text{pizzas with 10 possible toppings}$$

set of toppings $\{$ cheese, tomato, pickle, ... $\}$

subset of toppings $\longleftrightarrow$ pizza

$$\#\text{ of subsets of set size 10} = 2^{10}$$

$$\#\text{pizzas with } n \text{ toppings} = 2^n = \#\text{binary sequences length } n = \#\text{subsets of set size } n$$

Permutations

example: symbols A, B, C

how many 3 letter words using all symbols?

ABC BAC CAB
ACB BCA CBA

3 spots: $\underline{3} \cdot \underline{2} \cdot \underline{1} = 6$

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4 letter words: how many permutations (reorderings) of 4 letters?

$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 24 = 4!$ "4 factorial"

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 \cdot (5!) = 720$

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example: club 12 members

choose P, VP, Treas

how many ways?

$\frac{12}{P} \cdot \frac{11}{VP} \cdot \frac{10}{T} = 1320$

notation: ${}_{12}P_3$

(12 = # items, 3 = # picked (in order))

formula:

$${}_nP_r = \underbrace{\underline{n}\,\underline{(n-1)}\,\underline{(n-2)} \cdots \underline{(n-r+1)}}_{r \text{ spots}} \cdot \frac{(n-r)\cdots(1)}{(n-r)\cdots(1)}$$

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

Combinations (order does not matter)

example: Symbols A, B, C, D (4 total)
pick 2 — how many ways?

AB BC CD
AC BD
AD

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2 spots: $\frac{4 \cdot 3}{2!} = 6$ ← # ways to reorder the spots

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example: # ways to choose 3 items from 5

$\frac{5 \cdot 4 \cdot 3}{3!}$ ← # ways to reorder the 3 spots

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example: dinner party w/ 10 people
Cheers: clink glasses
how many clinks?
ways to choose 2 people from 10

$$\frac{10 \cdot 9}{2!} = 45$$

notation: $_{10}C_2 = \binom{10}{2}$ ← alternate notation

(10 = # items, 2 = # picked)

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example: $\binom{10}{3}$ = # ways to pick 3 items from 10

$$= \frac{10 \cdot 9 \cdot 8}{3!}$$

$$= \frac{\overset{5}{\cancel{10}} \cdot \overset{3}{\cancel{9}} \cdot 8}{\cancel{3} \cdot 2}$$

$$= 120$$

$\binom{10}{4}$ = "10 choose 4" = # ways to choose 4 items from 10

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!}$$

$$= \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \, 7}{\cancel{4} \cdot \cancel{3} \cdot 2}$$

$$= 210$$

$\binom{10}{1} = 10$ # of ways to pick 1 item from 10

$\binom{10}{0} = 1 = \binom{10}{10}$

$\binom{10}{9} = \binom{10}{1} = 10$

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formula:

$$_nC_r = \binom{n}{r} \quad \text{"n choose r"}$$

(n = # items, r = # picked)

$$= \underbrace{\frac{n \, (n-1)(n-2) \cdots (n-r+1)}{r \text{ spots}}}_{r \text{ spots}} \frac{(n-r) \cdots (1)}{r! \, (n-r) \cdots (1)}$$

ways to reorder the r spots

$$\boxed{_nC_r = \frac{n!}{r!\,(n-r)!}}$$