6.2 Binomial Theorem

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$
\begin{aligned}
(x+y)^3 &= (x+y)(x^2+2xy+y^2) \\
&= x^3 + 2x^2y + xy^2 \\
&\quad + x^2y + 2xy^2 + y^3 \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}
$$

$$(x+y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

$$
\begin{aligned}
(x+y)^n = {} & x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots \\
& + \boxed{\binom{n}{k}x^{n-k}y^k} + \dots + y^n
\end{aligned}
$$

WTF? why this formula?

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example:

expand $(a-2b)^4$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$$= \underline{1}(a)^4(-2b)^0 + \underline{4}(a)^3(-2b)^1 + \underline{6}(a)^2\boxed{(-2b)^2}$$
$$+\underline{4}(a)^1\boxed{(-2b)^3} + \underline{\ }(-2b)^4$$
$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$$

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example: find the a^3 term in $(a-2b)^7$

$$\binom{7}{3}(a)^3(-2b)^4$$
$$= 35 \cdot 16\, a^3b^4$$
$$= 560\, a^3b^4$$

or $\binom{7}{4}$

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$$

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why?

$$(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$$
$$= \binom{7}{0}x^7y^0 + \binom{7}{1}x^6y^1 + \binom{7}{2}x^5y^2 + \ldots$$

choose 2 spots from 7 to be y

or: $\binom{7}{5}$ choose 5 spots to be x

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

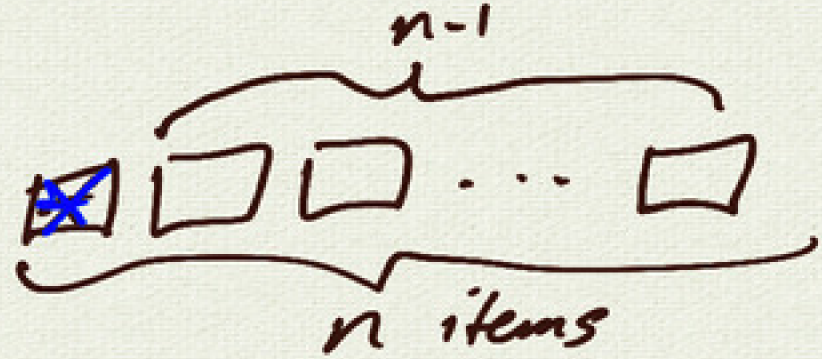
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \boxed{\binom{3}{2}} \quad \binom{3}{3}$$

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

recurrence relation

\# ways to choose k items from n total



choose k:

include special: $\binom{n-1}{k-1}$

don't include: $\binom{n-1}{k}$