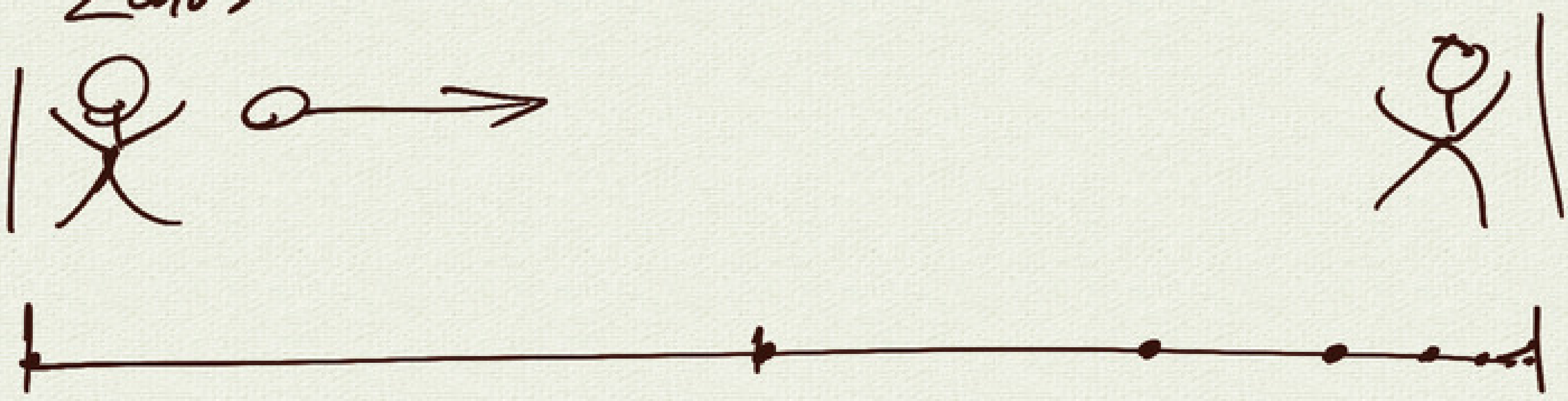
6.6 Series

Zeno's Paradox



$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$$

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notation:

S_n = sum of first n terms of a sequence $\{a_n\}$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_n = a_1 + \dots + a_n$$

$$= \sum_{i=1}^{n} a_i$$

end

dummy index

start

Sigma $\longleftrightarrow$ "sum"

<u>example</u>: sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ (geometric $r=\frac{1}{2}$)

$a_1 = \frac{1}{2}$

$$a_n = a_1 \cdot r^{n-1}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$\boxed{a_n = \left(\frac{1}{2}\right)^n}$$

n	$a_n = \left(\frac{1}{2}\right)^n$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$$S_3 = a_1 + a_2 + a_3$$

$$= \sum_{k=1}^{3} a_k$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{7}{8}$$

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example:

$$\sum_{i=3}^{5} a_i = a_3 + a_4 + a_5$$

$$= \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

$$= \frac{7}{32}$$

series = sum

5050

$$\underbrace{1+\underbrace{2+\underbrace{3+\cdots+98}_{101}+99}_{101}+100}_{101} = 50\cdot 101 = 5050$$

$$\underbrace{1+\underbrace{2+3+\cdots+98+99}_{\text{average } 101/2}+100}_{\text{average } \frac{101}{2}} = 100\cdot\left(\frac{101}{2}\right) = 5050$$

arithmetic sequence

$$\underbrace{a_1 \xrightarrow{+d} \underbrace{a_2 \xrightarrow{+d} a_3+\cdots+a_{n-2} \xrightarrow{+d} a_{n-1}} \xrightarrow{+d} a_n}_{\frac{a_1+a_n}{2}} = S_n$$

same average $\frac{a_1+a_n}{2}$

$$\boxed{S_n = \underbrace{\left(\frac{a_1+a_n}{2}\right)}_{\text{average}}\underbrace{n}_{\#\text{ terms}}}$$

some polynomial multiplication

$$(1-x)(1+x+x^2+\dots+x^{n-1})$$

$$= 1+\underbrace{\begin{matrix} x+x^2+\dots+x^{n-1} \\ -x-x^2-\dots-x^{n-1} \end{matrix}}_{\text{cancels}}-x^n$$

$$= 1-x^n$$

$$\Rightarrow \boxed{1+x+x^2+\dots+x^{n-1} = \frac{1-x^n}{1-x}}$$

challenge: verify with division

geometric:

$$a_1 \xrightarrow{\cdot r} a_2 \xrightarrow{\cdot r} a_3 \xrightarrow{\cdot r} a_4 \quad \dots$$

$a_n = a_1 r^{n-1}$ explicit formula

$$S_n = \sum_{k=1}^{n} a_k \qquad (a_1+a_2+\dots+a_n)$$

$$= \sum_{k=1}^{n} a_1 r^{k-1}$$

$$= a_1 + a_1 r + a_1 r^2 + \dots + a_1 r^{n-1}$$

$$= a_1 \underline{(1+r+r^2+\dots+r^{n-1})}$$

$$\boxed{S_n = a_1 \frac{(1-r^n)}{1-r}}$$

finite geometric sum (series)

example: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $\quad a_1 = \frac{1}{2}$, $r = \frac{1}{2}$

$S_2 = \frac{3}{4}$

$S_3 = \frac{7}{8}$

formula: $S_2 = \frac{\frac{1}{2}(1-(\frac{1}{2})^2)}{1-\frac{1}{2}} = 1 - \frac{1}{4} = \frac{3}{4}$

$\frac{1}{2}$

$S_3 = \frac{\frac{1}{2}(1-(\frac{1}{2})^3)}{1-\frac{1}{2}} = 1 - \frac{1}{8} = \frac{7}{8}$

$$S_n = 1 - \underbrace{\left(\tfrac{1}{2}\right)^n}_{\to 0 \text{ as } n\to\infty}$$

$S_n \to 1$ as $n \to \infty$

S_n = n^{th} partial sum

$$S_\infty = \lim_{n\to\infty} S_n = 1$$

$$S_\infty = \sum_{k=1}^{\infty} a_k \qquad \text{infinite sum}$$

$$\boxed{S_n = \frac{a_1(1-\boxed{r^n})}{1-r}}$$

if $|r|<1$:

$$S_n = \frac{a_1(1-r^n)}{1-r} \qquad r^n \to 0 \text{ as } n\to\infty$$

$$\boxed{S_\infty = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}}$$

infinite geometric series

example:

rational number $\Longleftrightarrow$ fraction

decimal expansion terminates or repeats

$$3.\boxed{456}\boxed{456}\overline{456}\ldots$$

$$= 3 + \underbrace{\frac{456}{1000} + \frac{456}{(1000)^2} + \frac{456}{(1000)^3} + \cdots}_{\text{geometric}}$$

geometric

$$a_1 = \frac{456}{1000} \qquad r = \frac{1}{1000}$$

$$S_\infty = \frac{a_1}{1-r} = \underbrace{\left(\frac{456}{1000}\right)}_{a_1}\frac{1}{1-\underbrace{\left(\frac{1}{1000}\right)}_{r}}$$

$$= \frac{456}{1000}\cdot\frac{1}{(999/1000)}$$

$$= \frac{456}{1000}\cdot\frac{1000}{999}$$

$$= \frac{456}{999}$$

$$\boxed{\begin{aligned} 3.\overline{456} &= 3 + \frac{456}{999} \\ &= 3\tfrac{456}{999} \end{aligned}}$$