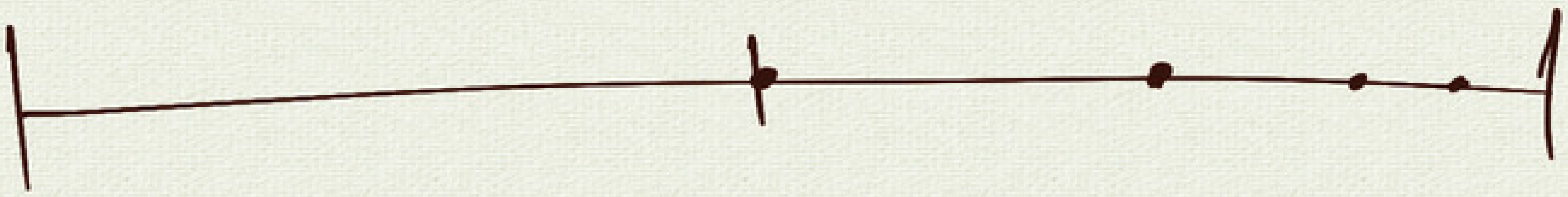
6.6 Series

Zeno's Paradox



$$\underset{\Uparrow a_1}{\frac{1}{2}} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

notation: $S_n = \underbrace{a_1 + a_2 + \dots + a_n}_{\text{sum of first } n \text{ terms}}$

n^{th} partial sum

example: $\{a_n\} = \cancel{\left(\frac{1}{2}\right)^{n-1}}$ $\left(= \left(\frac{1}{2}\right)^0, \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^2, \dots \right)$

1

$a_n = \left(\frac{1}{2}\right)^n$ $\left(= \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \dots \right)$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots$

$$S_2 = a_1 + a_2$$
$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_{11} = a_1 + a_2 + \dots + a_{11}$$
$$= \sum_{i=1}^{11} a_i$$

end

plug in

start

dummy index

summation notation

$\sum$ capital sigma "Sum"

$$\sum_{k=3}^{5} a_k = a_3 + a_4 + a_5$$

5050

$$\underbrace{1 + \underbrace{2 + \ldots + 99}_{101} + 100}_{101} = ?$$

$$50 \cdot 101 = 5050$$

$$\underbrace{1 + \underbrace{2 + 3 + \ldots + 98 + 99}_{\text{avg } 101/2} + 100}_{\text{avg. } 101/2} = 100 \cdot \frac{101}{2} = 5050$$

arithmetic sequence $a_1 \xrightarrow{+d} a_2 \xrightarrow{+d} a_3 \xrightarrow{+d} a_4$

$$S_n = \sum_{i=1}^{n} a_i$$

$$= \underbrace{a_1 \xrightarrow{+d} + \underbrace{a_2 + \ldots + a_{n-1}}} \xrightarrow{+d} + a_n}_{\text{avg. } \frac{a_1 + a_n}{2}}$$

same average

$$\boxed{S_n = \left(\frac{a_1 + a_n}{2}\right) \cdot n}$$

finite sum of arithmetic sequence

polynomial multiplication

$$(1-x)(1+x+x^2+\dots+x^{n-1})$$

$$= 1 + \boxed{\begin{array}{l} x+x^2+\dots+x^{n-1} \\ -x-x^2-x^3\dots-x^{n-1} \end{array}} - x^n$$

cancels

$$= 1-x^n$$

$$\Longrightarrow \boxed{1+x+x^2+\dots+x^{n-1} = \frac{1-x^n}{1-x}}$$

challenge: verify with division

---

geometric sequence:

$$a_1 \xrightarrow{\cdot r} a_2 \xrightarrow{\cdot r} a_3 \xrightarrow{\cdot r} a_4 \xrightarrow{\cdot r} a_5 \quad \dots$$

explicit: $a_n = a_1 \cdot r^{n-1}$

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} a_1 r^{k-1}$$

$$= a_1 + a_1 r + a_1 r^2 + \dots + a_1 r^{n-1}$$

$$= a_1(1+r+r^2+\dots+r^{n-1})$$

$$\boxed{S_n = a_1 \frac{(1-r^n)}{1-r}}$$

example: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

$$a_n = \left(\tfrac{1}{2}\right)^n$$

$$S_2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$

$$S_3 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8}$$

check formula: $\boxed{\begin{array}{l} a_1 = 1/2 \\ r = 1/2 \end{array}}$

$$S_2 = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^2\right)}{\underbrace{1-1/2}_{1/2}} = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

$$S_3 = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^3\right)}{1-1/2} = 1 - \tfrac{1}{8} = \tfrac{7}{8}$$

$$S_n = 1 - \underbrace{\left(\tfrac{1}{2}\right)^n}_{\to 0 \text{ as } n\to\infty}$$

$$S_n \to 1 \text{ as } n \to \infty$$

S_∞

$$S_\infty = \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$$

---

in general (for geometric)

$$S_n = a_1 \frac{(1-r^n)}{1-r}$$

$\to 1$ when $|r|<1$
"r is a fraction"

$$\boxed{S_\infty = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}}$$

when $|r|<1$

example: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

$$a_1 = \tfrac{1}{2} \quad r = \tfrac{1}{2}$$

$$S_\infty = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n$$

$$= \frac{a_1}{1-r} = \frac{(1/2)}{1-(1/2)} = 1$$

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots = 1$$

"illegal" infinite polynomial

$$(1-x)(1+x+x^2+\dots)$$

$$= 1 + x + x^2 + x^3 + \dots$$
$$\quad - x - x^2 - x^3 - \dots$$

cancel

$$= 1$$

$$\boxed{1+x+x^2+\dots = \frac{1}{1-x}}$$

$$a_1 + a_1 r + \dots = \frac{a_1}{1-r}$$
$$a_1(1+r+r^2+\dots)$$

---

example: $5.123\overline{123}$ ← rational

⟹ fraction

$$5.\overline{123} = 5 + \frac{123}{1000} + \frac{123}{(1000)^2} + \frac{123}{(1000)^3} + \dots$$

geometric

$a_1 = \frac{123}{1000}$ $\qquad r = \frac{1}{1000}$

$$S_\infty = \frac{a_1}{1-r} = \frac{\left(\frac{123}{1000}\right)}{1-\left(\frac{1}{1000}\right)}$$

$$= \frac{\left(\frac{123}{1000}\right)}{\left(\frac{999}{1000}\right)}$$

$$= \frac{123}{999}$$

$$\boxed{5.\overline{123} = 5 + \frac{123}{199}}$$