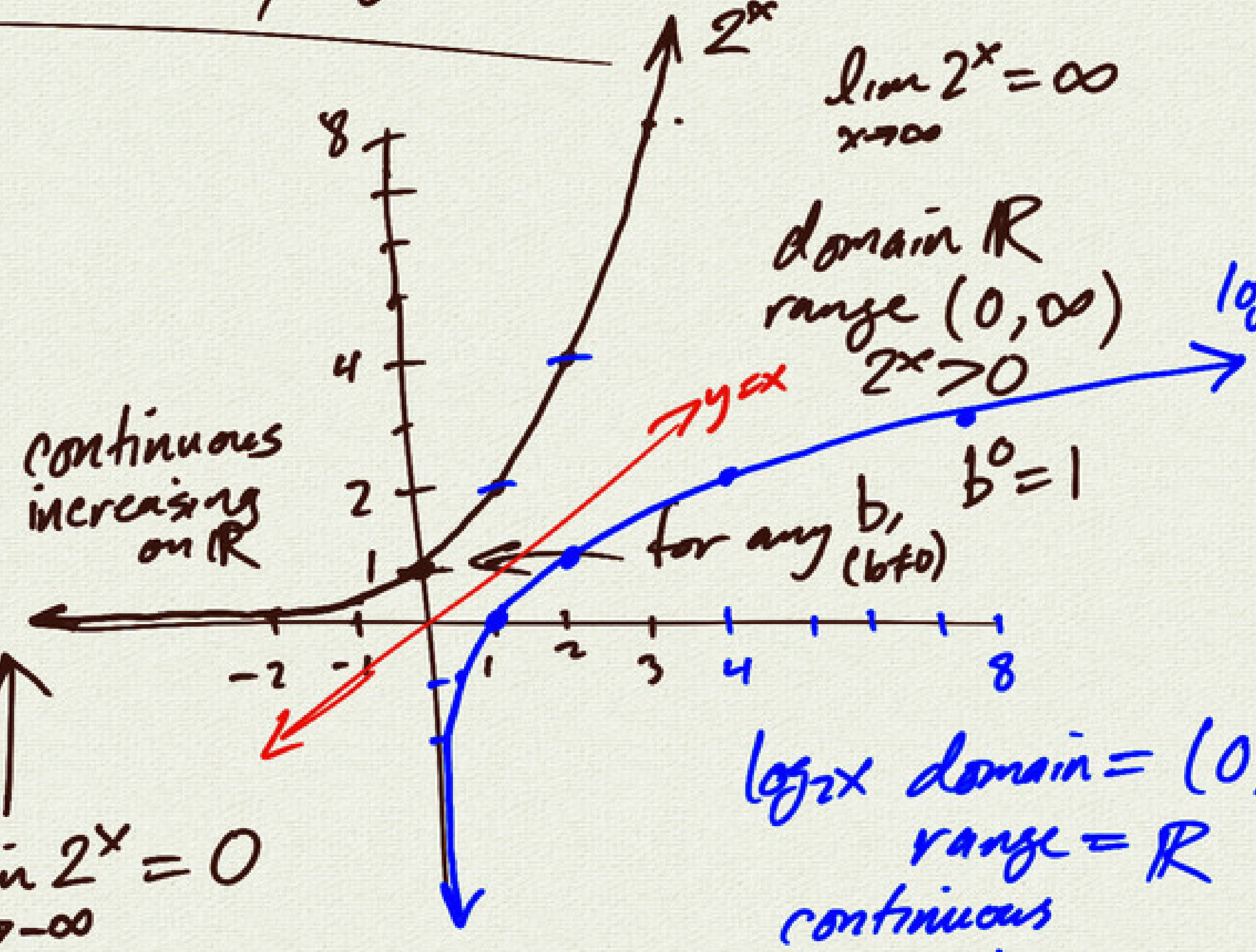
7.1 Exponential/Logarithm

$y = 2^x$

x	2^x
0	1
1	2
2	4
3	8
-1	$1/2$
-2	$1/4$

$\log_2$



$\lim_{x\to\infty} 2^x = \infty$

domain $\mathbb{R}$
range $(0,\infty)$
$2^x > 0$

$b^0 = 1$ for any b, $(b \neq 0)$

continuous increasing on $\mathbb{R}$

$\lim_{x\to-\infty} 2^x = 0$

$\log_2 x$ domain $= (0,\infty)$
range $= \mathbb{R}$
continuous
increasing

$2 \xrightarrow{10^x} 100$

$3 \longrightarrow 1000$

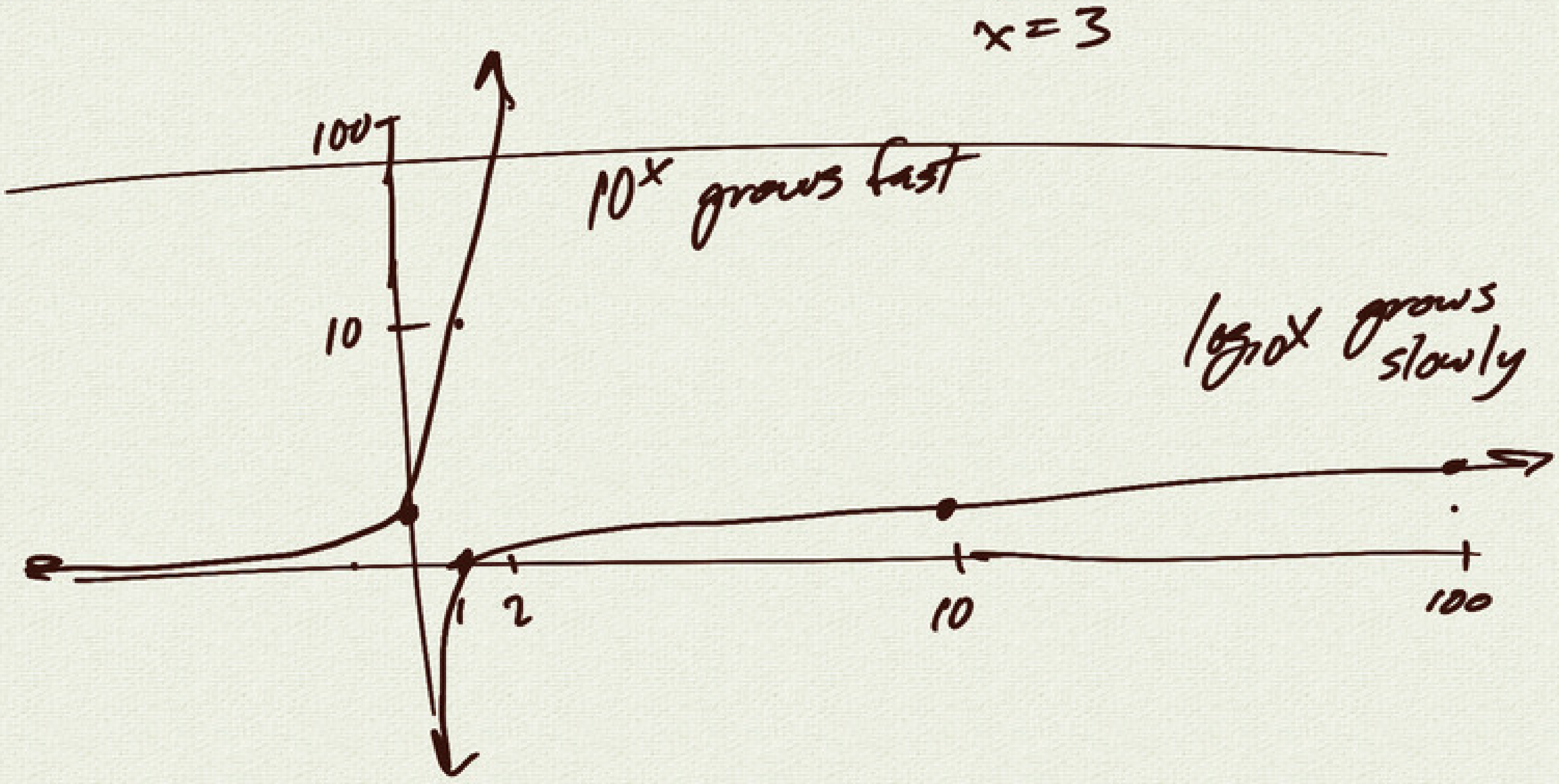
$3 \xleftarrow{\log_{10}} 1000$

10^x and $\log_{10} x$ are inverse functions

2^x and $\log_2 x$ are inverses

$x = \log_2 8 \iff 2^x = 8$

$x = 3$



10^x grows fast

$\log_{10} x$ grows slowly

exponential properties:

$$b^x b^y = b^{x+y}$$
$$(b^x)^y = b^{xy}$$

$$\underbrace{b \cdots b}_{x}\,\underbrace{b \cdots b}_{y} = b^{x+y}$$

$$\underbrace{\underbrace{b \cdots b}_{x}\,\underbrace{b \cdots b}_{x} \cdots \underbrace{b \cdots b}_{x}}_{y} = (b^x)^y = b^{xy}$$

$$2 \xrightarrow{10^x} 100 = 10^2$$
$$3 \longrightarrow 1000 = 10^3$$
$$2+3=5 \longrightarrow 10^5 = 10^2 \cdot 10^3$$

addition $\xleftarrow{\log_{10} x}$ multiplication

properties:

$$\boxed{\begin{aligned} \log_b(xy) &= \log_b x + \log_b y \\ \log_b(x^n) &= n \log_b x \end{aligned}}$$

base change

compute:

$$\log_b y = x \quad (\text{find } x)$$

$$\Updownarrow$$

$$y = b^x$$

$$\ln y = \ln(b^x)$$

$$\ln y = x \ln b$$

$$x = \frac{\ln y}{\ln b}$$

$$\boxed{\log_b y = \frac{\ln y}{\ln b} = \frac{\log_{10} y}{\log_{10} b}}$$

example: $\log_5 23 = \dfrac{\ln 23}{\ln 5} \quad \left(= \dfrac{\log_{10} 23}{\log_{10} 5} \right)$

$$y = b^x \longleftrightarrow x = \log_b y$$

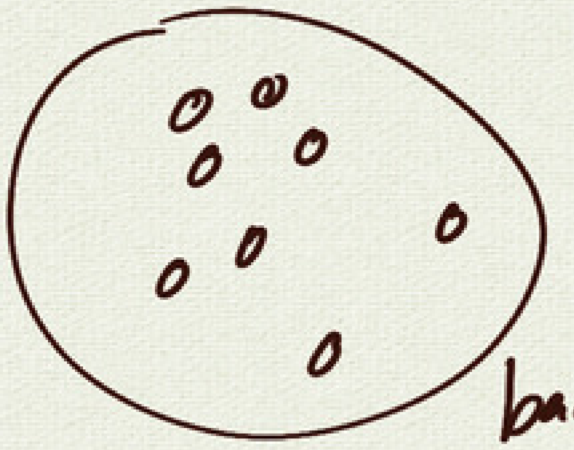
$$y = 2^x \longrightarrow x = \log_2 y$$

$$y = e^x \longleftrightarrow x = \log_e y$$
$$= \ln y$$

$\Uparrow$ natural log

$$2 < e < 10$$
$$\approx 2.718$$

example: population growth



bacteria

growth rate proportional to population size

initial population $P_0 = 10000$

doubling time 4 hours

model population $\boxed{P(t) = P_0 \cdot 2^{t/4}}$

$= 10000\left(2^{t/4}\right)$

doubling time

t	$P(t)$
0	$10000 = P_0$
4	$20000 = 10000 \cdot 2$
8	$40000 = 10000 \cdot 2^2$
12	$80000 = 10000 \cdot 2^3$

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could model with e^t:

$P(t) = P_0 e^{kt}$ (find P_0, k)

10000

find k: $P(0) = 10000 = P_0 e^{k \cdot 0} \longrightarrow P_0 = 10000$

$P(4) = 20000 = 10000 e^{k4}$

$2 = e^{4k}$

$\ln 2 = 4k \leftarrow \ln(e^{4k}) = 4k$

$\ln(e^x) = x$

$k = \frac{\ln 2}{4}$

$e^{\ln x} = x$

$\boxed{P(t) = 10000 e^{\frac{\ln 2}{4} t}}$

check: $P(4) = 10000 e^{\frac{\ln 2}{4} \cdot 4}$

$= 1000(e^{\ln 2})$

$= 20000$ ✓