9.3 Exponential/Logarithm

1. polynomials
2. trig
3. exp/log $\Longleftarrow$

$$x^3 \xrightarrow{\frac{d}{dx}} 3x^2$$
$$x^2 \longrightarrow 2x$$
$$x^1 \longrightarrow 1$$
$$x^0 \longrightarrow 0 \quad \Longleftarrow \text{where is } x^{-1}?$$
$$x^{-1} \longrightarrow -x^{-2}$$
$$x^{-2} \longrightarrow -2x^{-3}$$

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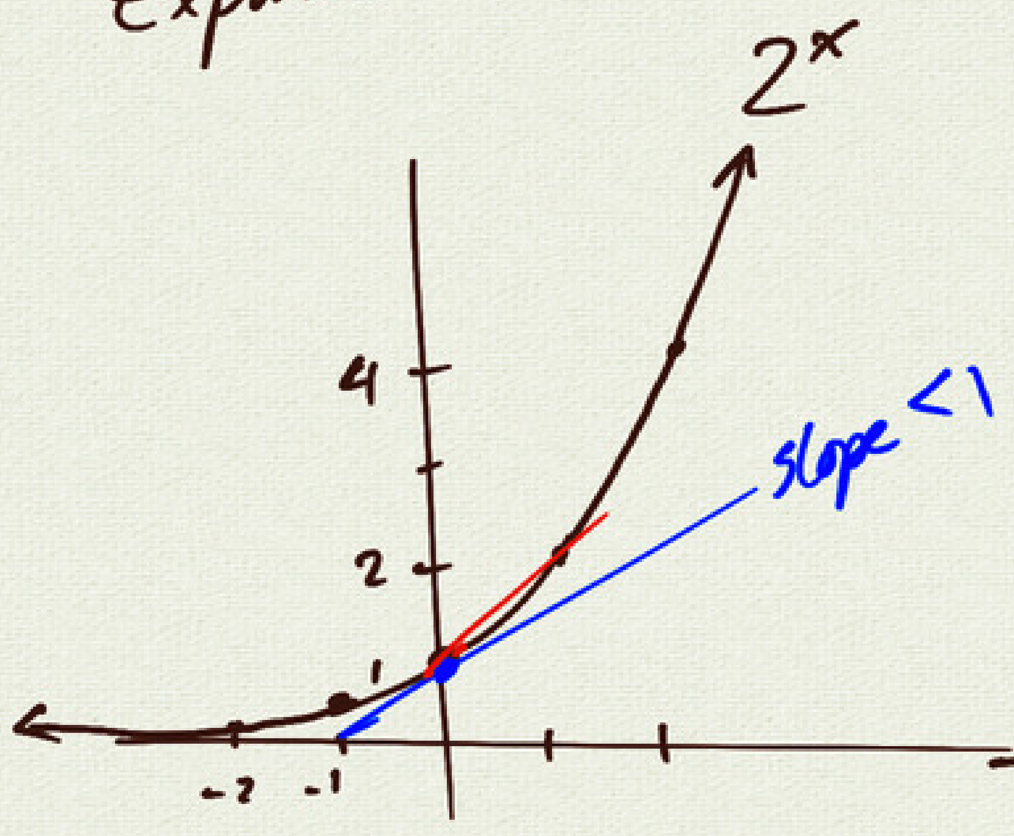
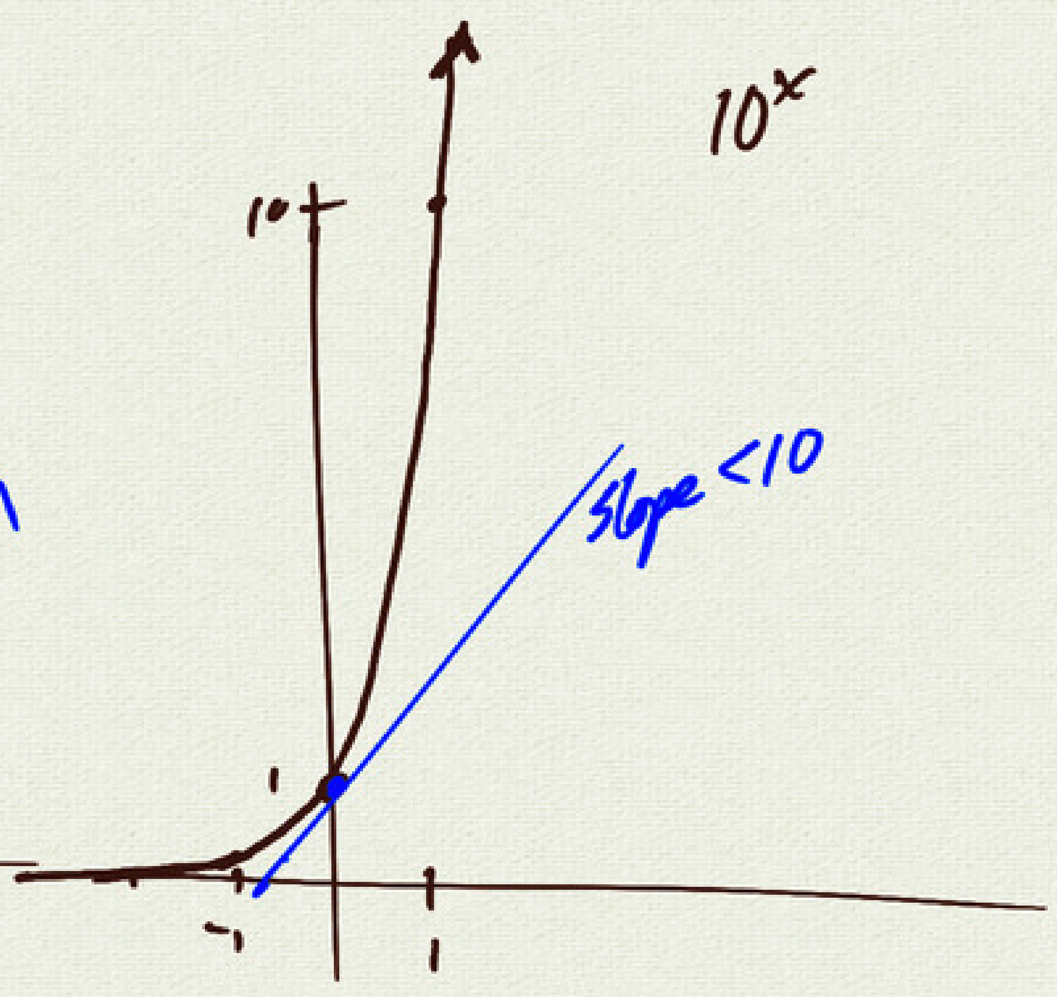
exponential functions





$f(x) = 2^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h}$$
$$= \lim_{h\to 0} \frac{2^x 2^h - 2^x}{h}$$
$$= 2^x \boxed{\lim_{h\to 0} \frac{2^h - 1}{h}} \, ? \quad \Leftarrow f'(0)$$

$$\boxed{2^{x+h} = 2^x 2^h}$$

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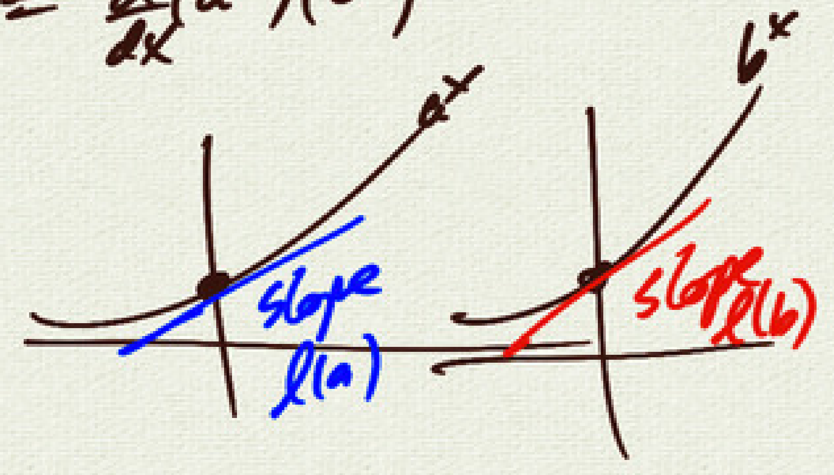
for some function a^x, let $\ell(a)$ = slope at 0 $= \frac{d}{dx}(a^x)(0)$



$f(x) = a^x \qquad g(x) = b^x$

$f(0) = 1 \qquad g(0) = 1$

$f'(0) = \ell(a) \qquad g'(0) = \ell(b)$

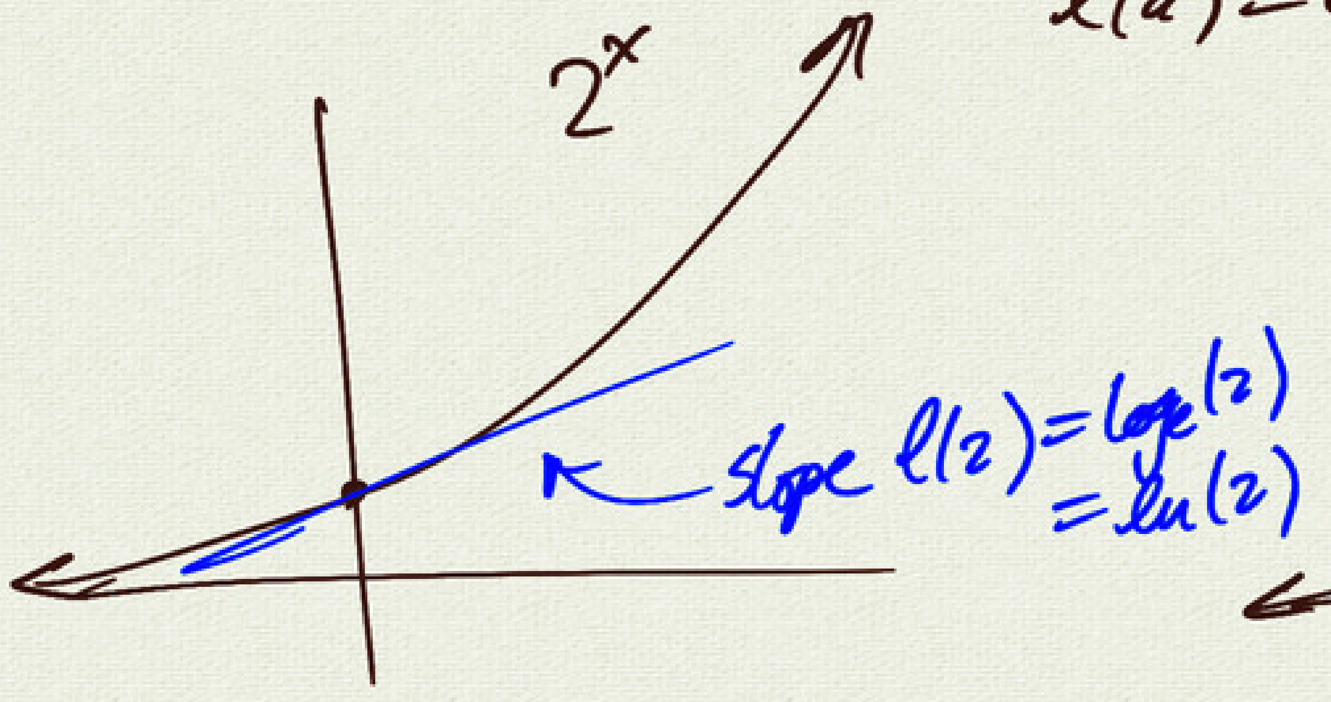
product: $(fg)(x) = a^x b^x$
$= (ab)^x$ exponential, base ab

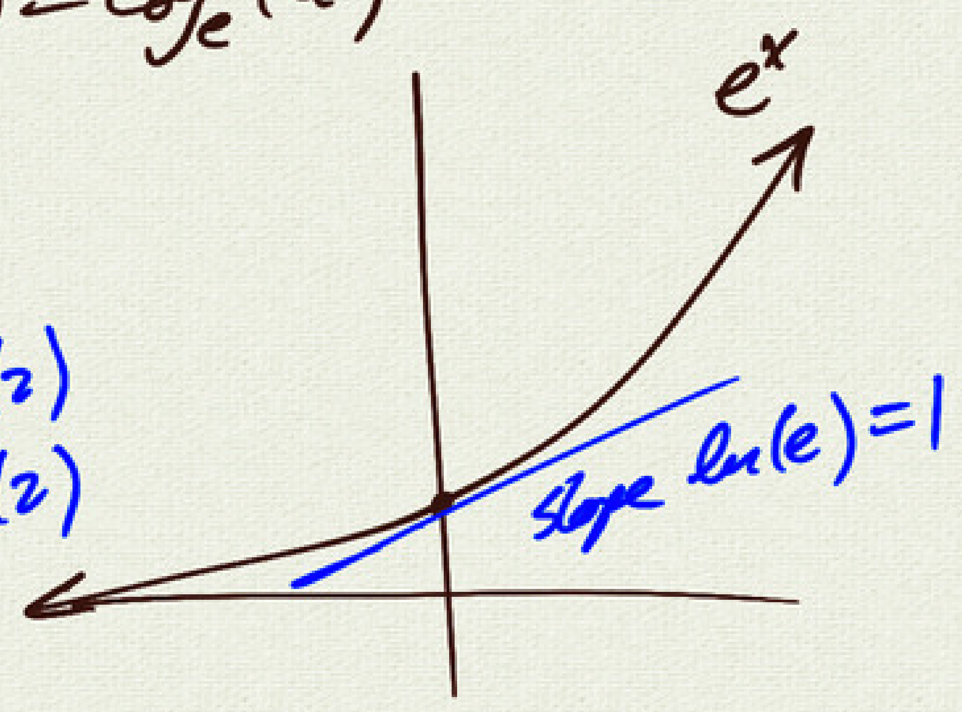
$\ell(ab) = (fg)'(0) = f'(0)g(0) + f(0)g'(0)$ product rule
$= \ell(a)\cdot 1 + 1\cdot \ell(b)$

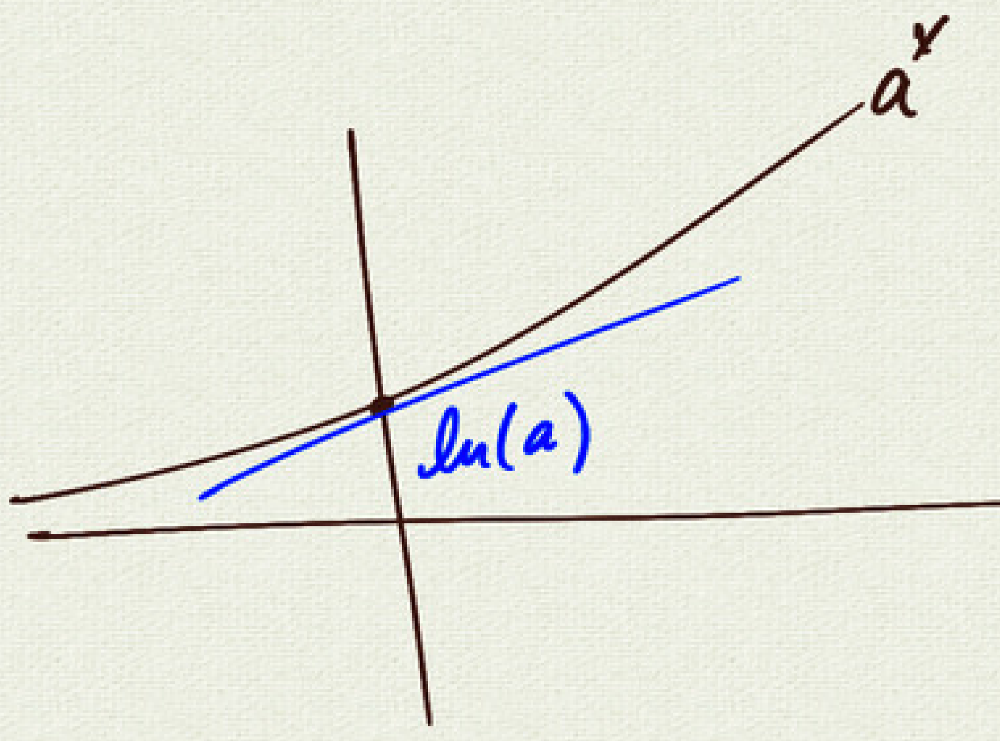
$$\boxed{\ell(ab) = \ell(a) + \ell(b)} \text{ logarithm}$$

$\Rightarrow$ base e

$\ell(a) = \log_e(a)$







$$\frac{d}{dx}(a^x)(0) = \lim_{h\to 0} \frac{a^{0+h} - a^0}{h}$$
$$= \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$\boxed{\lim_{h\to 0} \frac{a^h - 1}{h} = \ln(a)}$$

special limits

$(a = e)$
$$\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$$

$f(x) = e^x$

$\Longrightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h}$$
$$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$
$$= e^x \underbrace{\boxed{\lim_{h\to 0} \frac{e^h-1}{h}}}_{=1}$$
$$= e^x$$

$$\boxed{\frac{d(e^x)}{dx} = e^x}$$

special limits:

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1$$

$$\lim_{h\to 0} \frac{a^h-1}{h} = \ln(a)$$

$g(x) = a^x \longrightarrow$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$
$$= \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$
$$= a^x \underbrace{\boxed{\lim_{h\to 0} \frac{a^h-1}{h}}}_{\ln(a)}$$

$$\Longrightarrow \boxed{\frac{d(a^x)}{dx} = a^x \ln(a)}$$

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examples:

$f(x) = e^{\sin x} \Longrightarrow f'(x) = e^{\sin x} \cdot \cos x$

$g(x) = e^{x^3+x^2} \longrightarrow g'(x) = e^{x^3+x^2} \cdot (3x^2+2x)$

$h(x) = \tan(e^x + x^2) \Rightarrow h'(x) = \sec^2(e^x+x^2) \cdot (e^x+2x)$

$k(x) = 2^x \Longrightarrow k'(x) = 2^x \ln 2$

$l(x) = 2^{\cos x} \Longrightarrow l'(x) = 2^{\cos x} \ln 2 \cdot (-\sin x)$

$y = \ln x = \log_e x \longleftrightarrow e^y = x$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

there it is!

x^{-1}

$y = \log_a x \longleftrightarrow a^y = x$

$$a^y \ln a \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

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examples:

$$f(x) = \ln(\sin x + x^2)$$

$$\longrightarrow f'(x) = \frac{1}{\sin x + x^2} \cdot (\cos x + 2x)$$

$$g(x) = \log_{10}(\cot x - x^5)$$

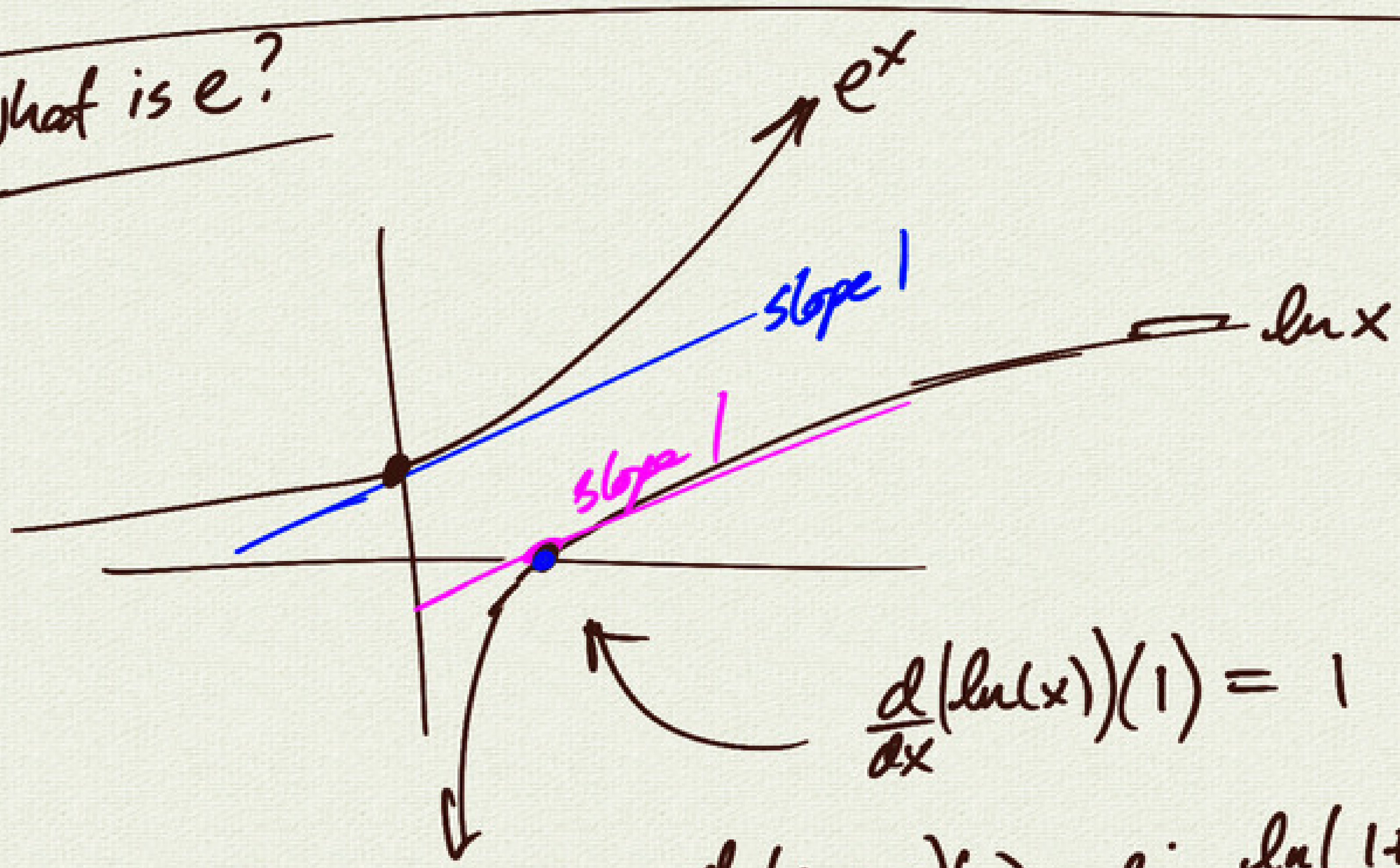
$$\longrightarrow g'(x) = \frac{1}{(\cot x - x^5)\ln 10} \cdot (-\csc^2 x - 5x^4)$$

Summary:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln a \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

what is e?



$$\frac{d}{dx}(\ln(x))(1) = 1$$

$$\frac{d}{dx}(\ln(x))(1) = \lim_{h \to 0} \frac{\ln(1+h) - \underbrace{\ln(1)}_{=0}}{h}$$

$$1 = \lim_{h \to 0} \frac{\ln(1+h)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \ln(1+h)$$

$$1 = \lim_{h \to 0} \ln(1+h)^{1/h}$$

$$e = \lim_{h \to 0} (1+h)^{1/h} \qquad h \approx \frac{1}{n}$$

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

$\ln(x^n)$
$= n \ln x$

$$\left(1 + \frac{1}{n}\right)^n = 1^n + \binom{n}{1} 1^{n-1} \left(\frac{1}{n}\right)^1 + \binom{n}{2} 1^{n-2} \left(\frac{1}{n}\right)^2 + \ldots$$

$$= 1 + \frac{n}{n} \cdot 1 + \frac{(n)(n-1)}{2} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} + \ldots$$

$$\Longrightarrow e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

$$\approx 2.71828\ldots$$