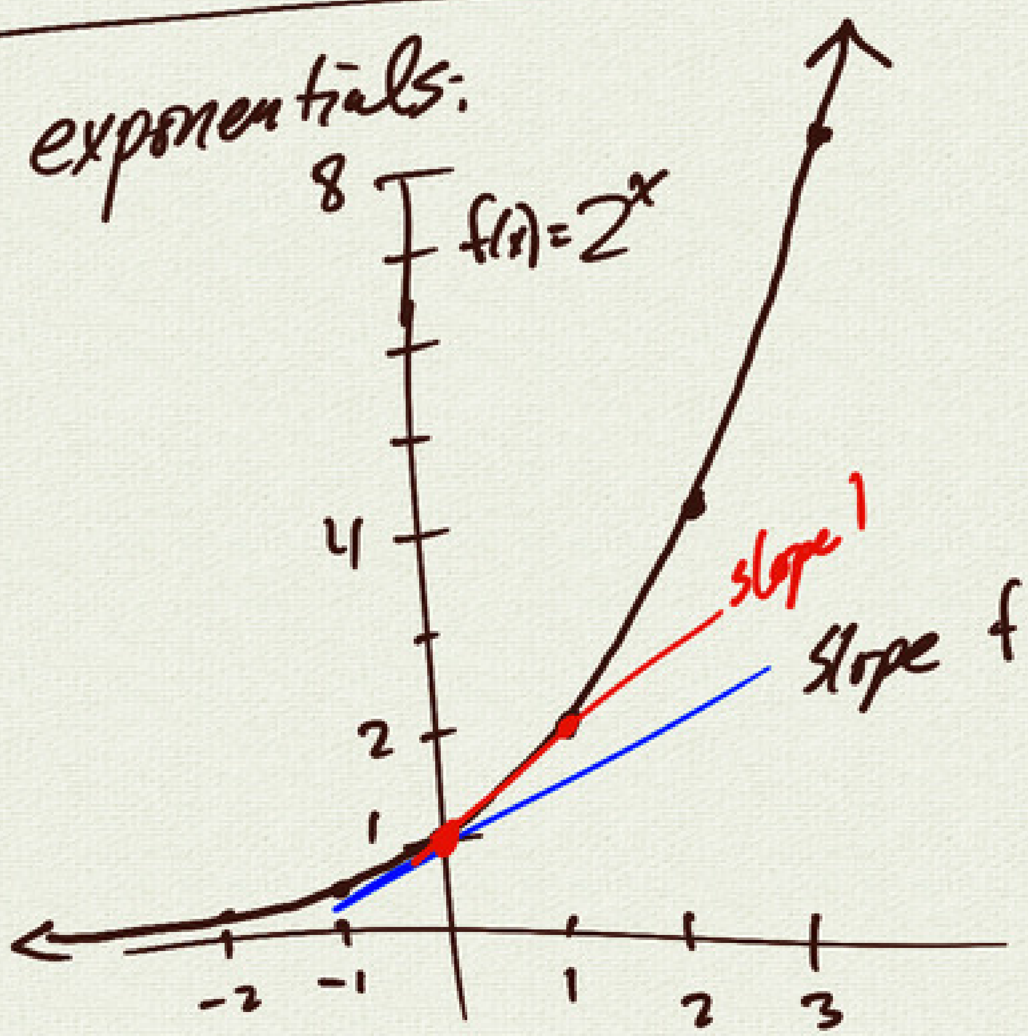
9.3 Exponential/Logarithm

1. polynomials
2. trig functions
3. exp/log

$$x^3 \xrightarrow{\frac{d}{dx}} 3x^2$$
$$x^2 \longrightarrow 2x^1$$
$$x^1 \longrightarrow 1x^0$$
$$x^0 \longrightarrow 0$$
$$x^{-1} \longrightarrow -1\cdot x^{-2}$$
$$x^{-2} \longrightarrow -2x^{-3}$$

where is x^{-1}?

exponentials:



slope $f'(0) = \lim\limits_{h\to 0} \dfrac{2^{0+h} - 2^0}{h}$

$$= \lim_{h\to 0} \frac{2^h - 1}{h} = \, ?$$

(looks like slope < 1)

$$f(x) = e^x$$

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right)$$

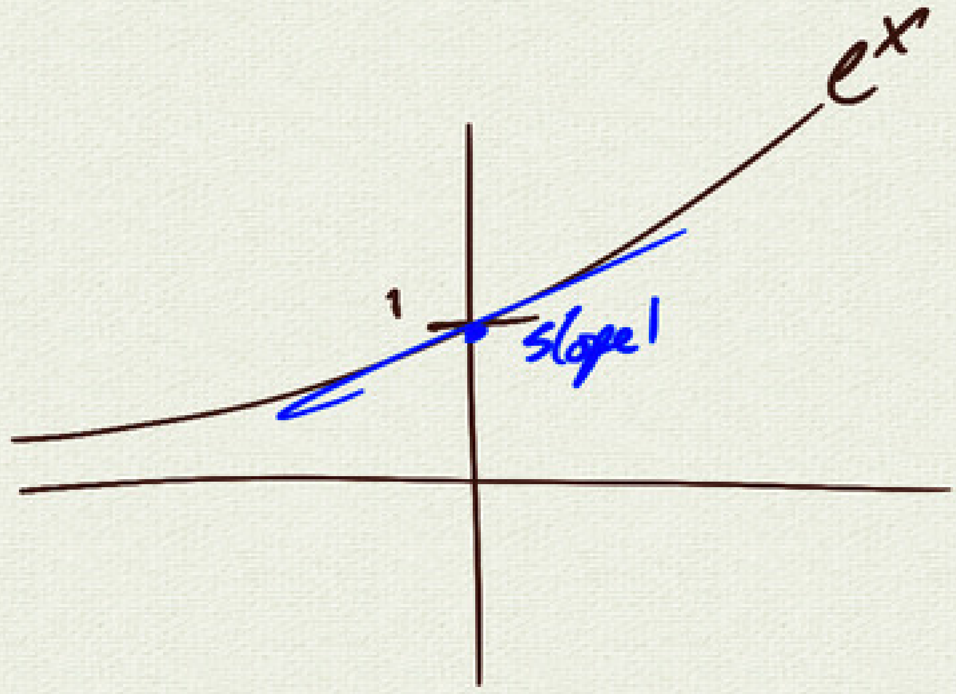
$$= e^x \boxed{\lim_{h\to 0} \frac{e^h - 1}{h}}$$

$$\boxed{\frac{d(e^x)}{dx} = e^x}$$

special limits:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln(a)$$



$$g(x) = a^x$$

$$g'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \boxed{\lim_{h\to 0} \frac{a^h - 1}{h}}\ \ln(a)$$

$$\Rightarrow \boxed{\frac{d(a^x)}{dx} = a^x \ln a}$$

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examples:

$$f(x) = e^{\sin x} \Rightarrow f'(x) = e^{\sin x}(\cos x)$$

$$g(x) = \cos(x^5 + 5e^x) \Rightarrow g'(x) = -\sin(x^5 + 5e^x) \cdot (5x^4 + 5e^x)$$

$$h(x) = 2^x + x^2 + 10^{\sin x}$$

$$\Rightarrow h'(x) = 2^x \ln 2 + 2x + 10^{\sin x} \ln 10 \cdot (\cos x)$$

Logarithms:

$y = \ln x \iff e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$ there it is!

$y = \log_a x \iff a^y = x$

$a^y \ln a \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$

$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$

Summary:

$$\frac{d(e^x)}{dx} = e^x \qquad \frac{d(\ln x)}{dx} = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln a \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

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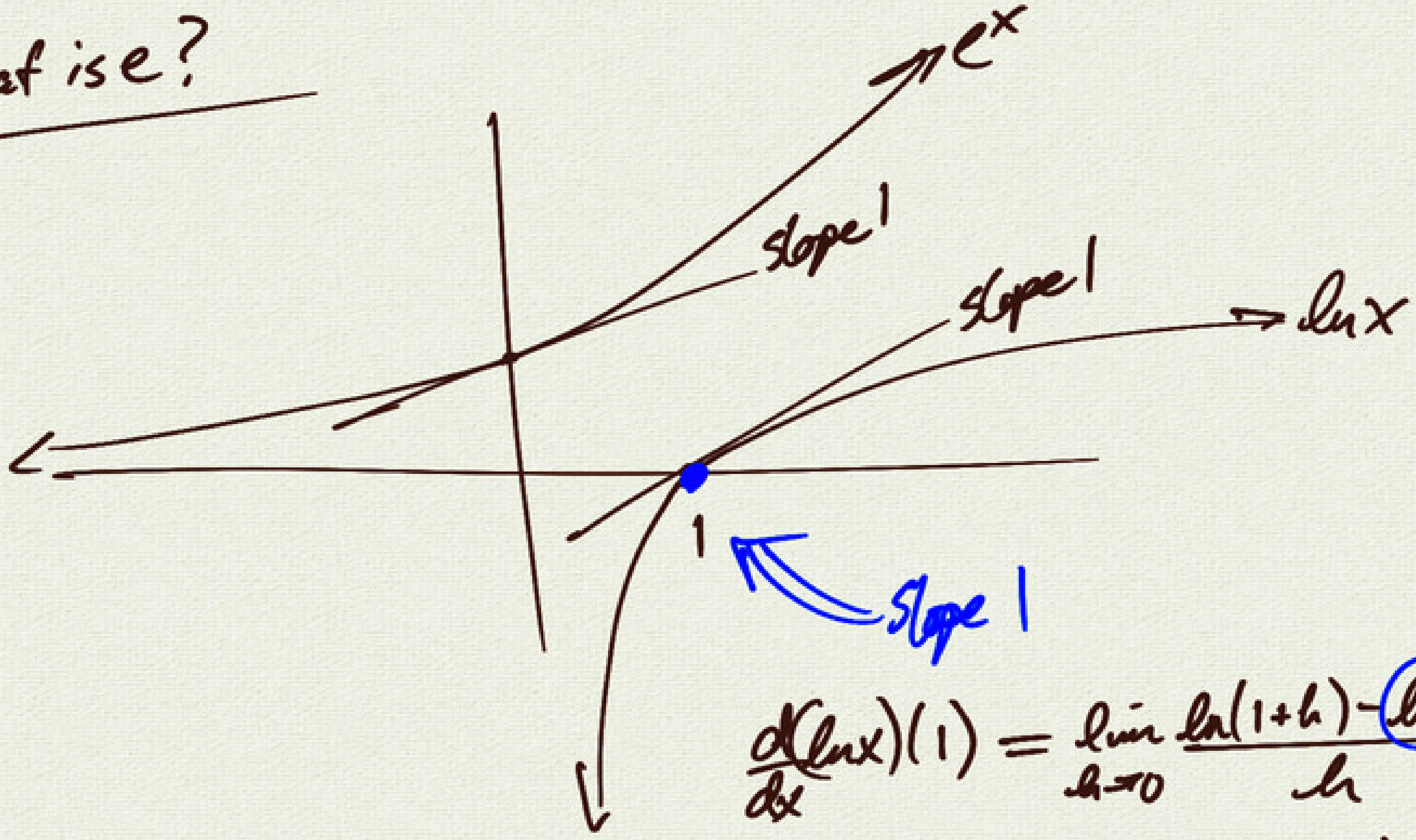
examples:

$$f(x) = \ln(\sin x) \implies f'(x) = \frac{1}{\sin x} \cdot \cos x \quad (= \cot x)$$

$$g(x) = \log_{10}(x^5 + e^x)$$

$$\longrightarrow g'(x) = \frac{1}{(x^5 + e^x)\ln 10} \cdot (5x^4 + e^x)$$

what is e?



$$\frac{d}{dx}(\ln x)(1) = \lim_{h\to 0}\frac{\ln(1+h) - \overset{0}{\ln 1}}{h}$$

$$1 = \lim_{h\to 0}\frac{1}{h}\ln(1+h) \qquad\qquad \ln x^n = n\ln x$$

$$1 = \lim_{h\to 0}\ln(1+h)^{1/h}$$

$$e = \lim_{h\to 0}(1+h)^{1/h} \qquad\qquad h \approx \frac{1}{n} \quad n\to\infty$$

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

$$\left(1+\frac{1}{n}\right)^n = 1^n + \binom{n}{1}1^{n-1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}1^{n-2}\left(\frac{1}{n}\right)^2 + \dots$$

$$= 1 + \frac{n}{n} + \underbrace{\frac{n(n-1)}{2\,n^2}}_{\to 1} + \underbrace{\frac{n(n-1)(n-2)}{3!\,n^3}}_{\to 1} + \dots$$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$\approx 2.71828\dots$$