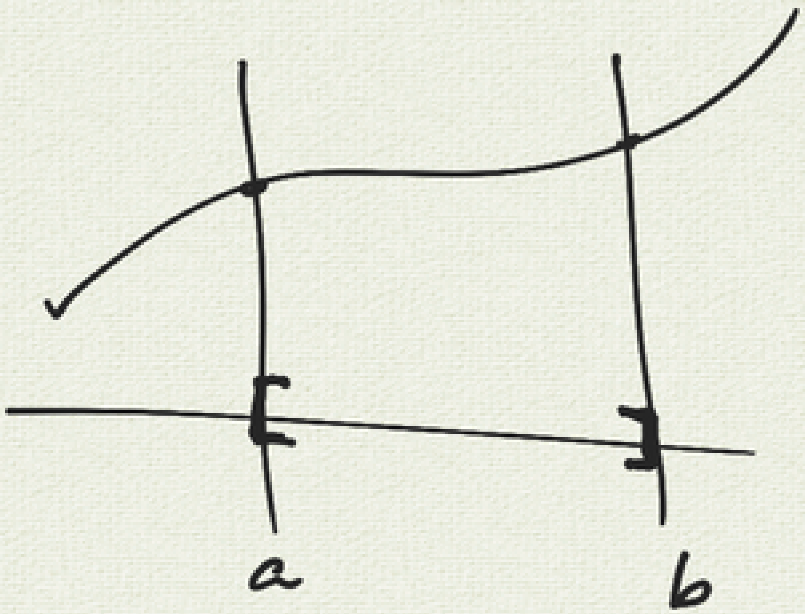
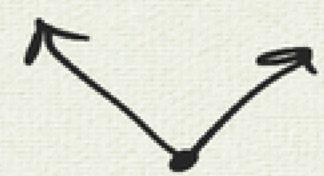
10.2 Mean Value Theorem

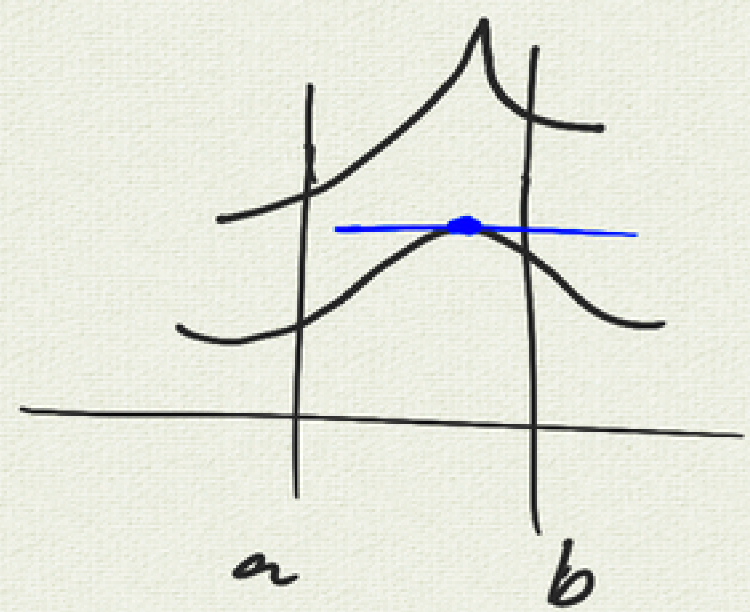
Extreme Value Theorem:
if f is continuous on a finite closed interval $[a,b]$, then it has an absolute max/min on the interval.



$\Rightarrow$ check ① end points
② critical points $f'=0$ or f' DNE

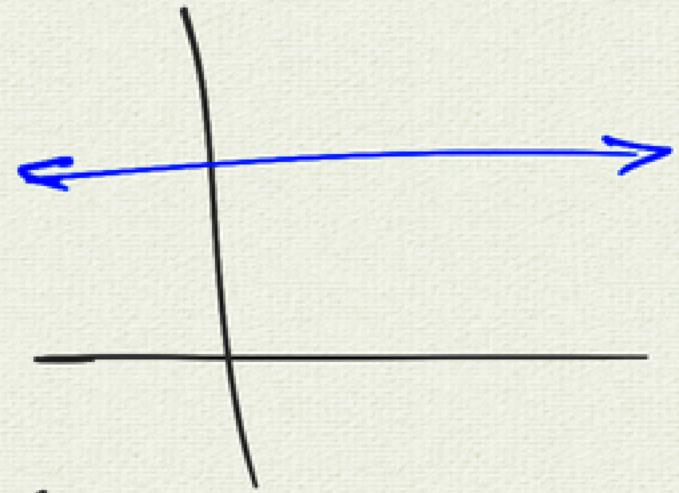


suppose f is continuous on $[a,b]$,
f has a local max at $x=c$ and $f'(c)$ exists,
then $f'(c)=0$



observation:

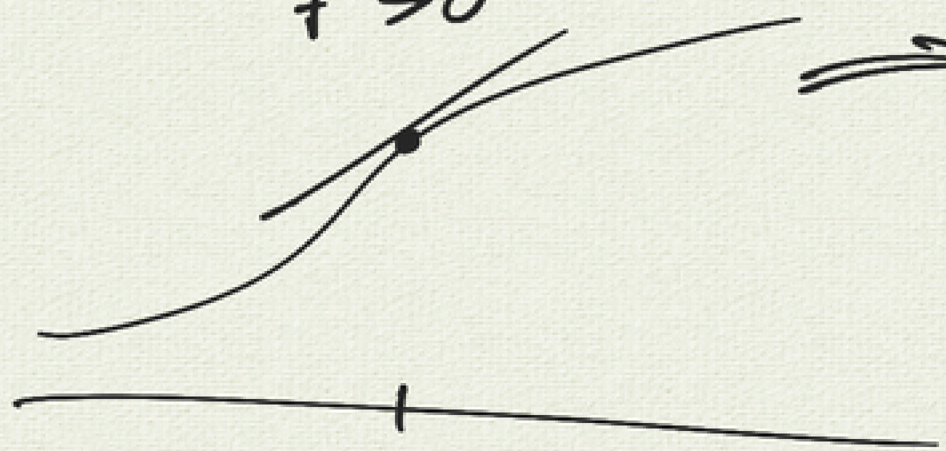
if $f(x) = \text{const}$, then $f'(x) = 0$



is converse true?

if $g'(x) = 0$, then $g(x) = \text{const}$

$f' > 0$



$\Longrightarrow$ f increasing?

$h(x) = x^2 \Longrightarrow h'(x) = 2x$

"converse"

$g'(x) = 2x \Longrightarrow g(x) = x^2$

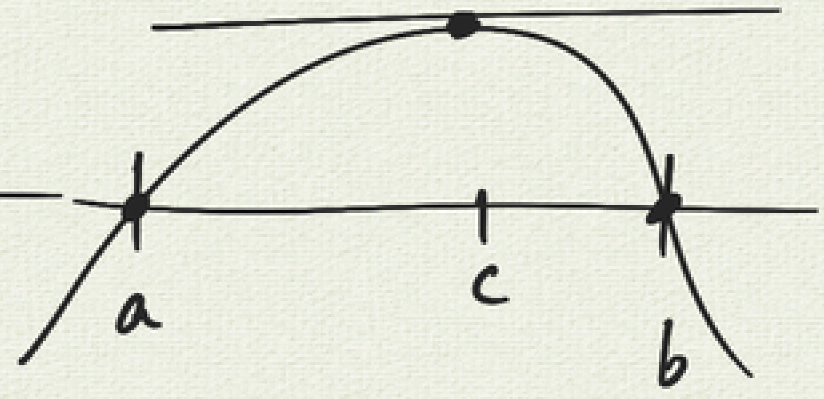
$x^2 + \text{const.}$

Rolle's Theorem

Suppose f is continuous on $[a,b]$

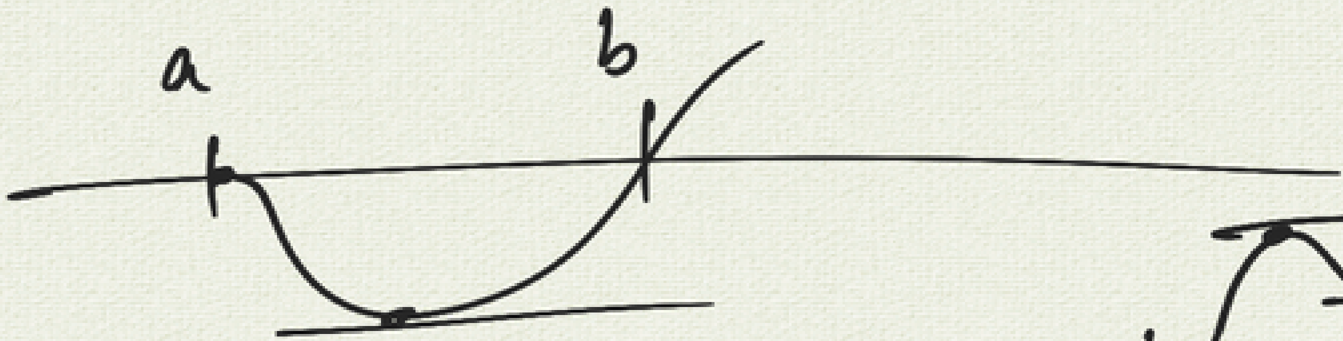
$f(a) = 0 = f(b)$

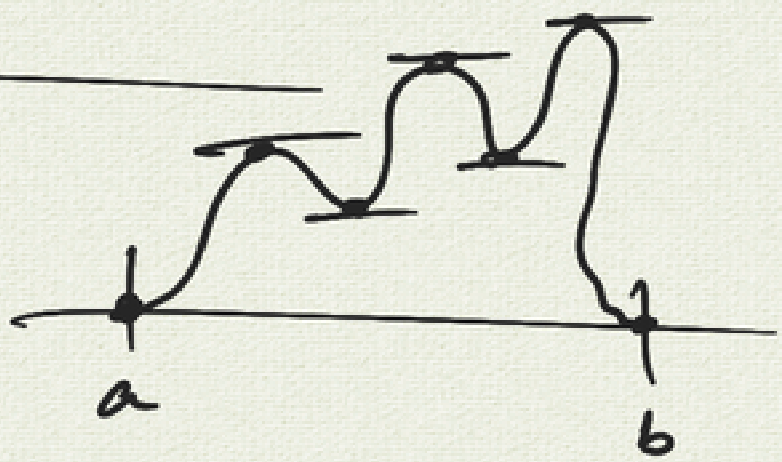
f differentiable on (a,b)



then $\exists c$ in $[a,b]$ such that

$$f'(c) = 0$$





Mean Value Theorem

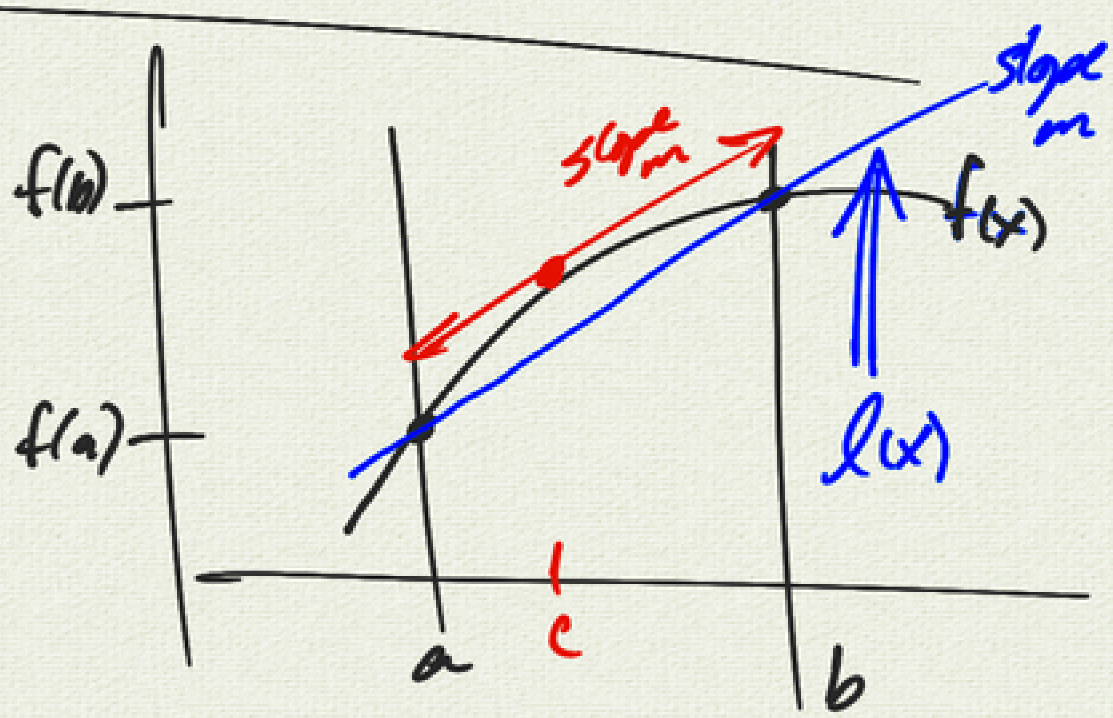
f continuous on $[a,b]$

f differentiable on (a,b)

let $m = \dfrac{f(b)-f(a)}{b-a}$

Then $\exists c$ in (a,b) such that

$$f'(c) = m$$



Proof: apply Rolle's to $f(x) - l(x)$

Corollary 1

Suppose $f'(x) = 0$ (on an interval, possibly $\mathbb{R}$)

Then $f = \text{const}$.

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take any a, b.

Mean value theorem $\Rightarrow$ $\exists c$ in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\underbrace{f'(c)}_{0}(b-a) = f(b) - f(a) = 0$$

$$\Rightarrow f(b) = f(a)$$

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f is constant

$\boxed{\text{Corollary 2}}$ if $f' = g'$,

then $f = g + \text{const}$.

Proof: consider $h = f - g$

$h' = f' - g' = 0$

$h = \text{const} \rightarrow f - g = \text{const}$.

$f = g + \text{const}$.

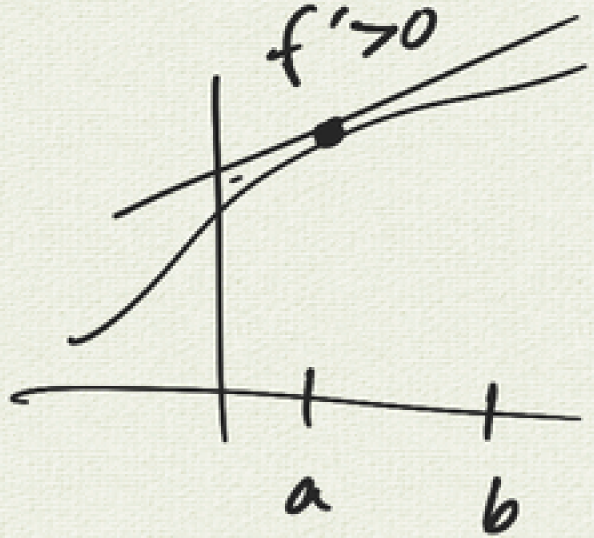
$g'(x) = 2x \Rightarrow g(x) = \underbrace{x^2 + C}_{\text{anti-derivative of } 2x}$

notation: $\int 2x\,dx = x^2 + C$

Corollary 3

if $\underbrace{f'(x) > 0}$ (on an interval) $\Rightarrow$ increasing

then f is increasing



Proof: take any a, b

then $\exists c$ in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\underbrace{f'(c)}_{+}\underset{+}{(b-a)} = \underbrace{f(b) - f(a)}_{+} \Rightarrow \boxed{\begin{array}{c} b > a \\ \Downarrow \\ f(b) > f(a) \end{array}}$$ increasing