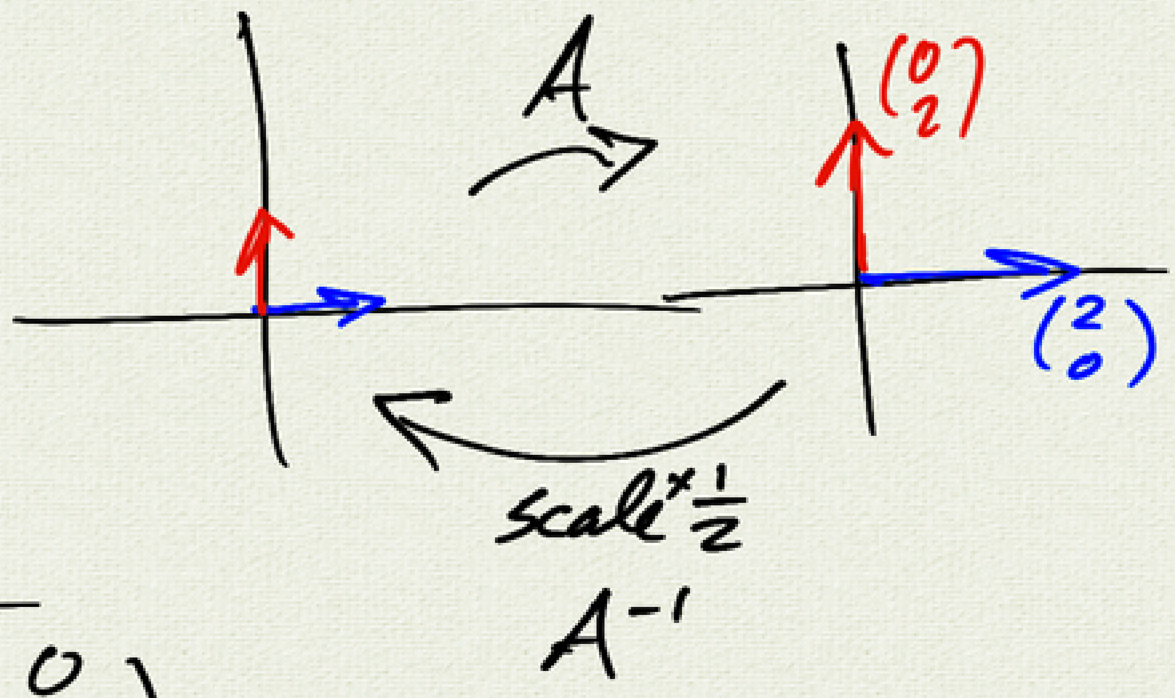
3.8 Matrix inverses and determinants

examples:

$A = \text{scale} \times 2$

$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$= 2I$



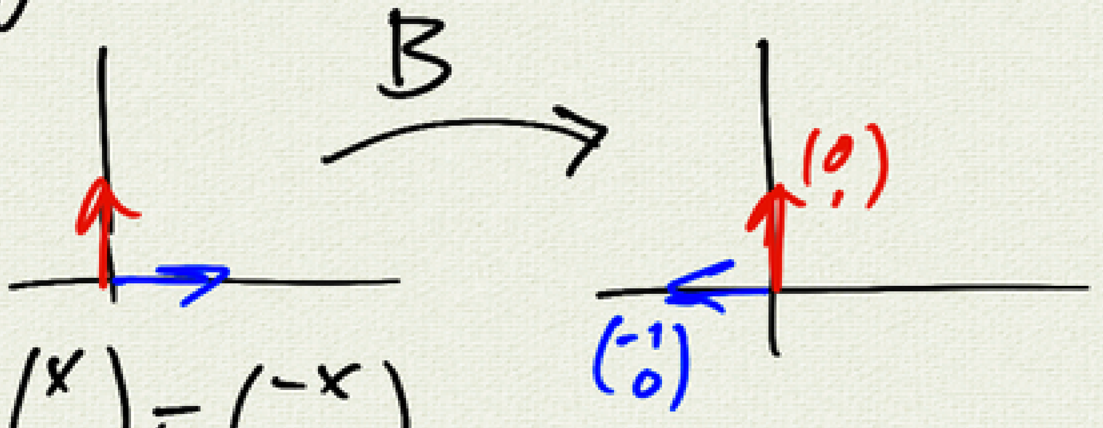
$A^{-1} = \frac{1}{2} I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$AA^{-1} = \begin{pmatrix} \boxed{2 \quad 0} \\ 0 \quad 2 \end{pmatrix} \begin{pmatrix} \boxed{1/2 \atop 0} & \begin{matrix} 0 \\ 1/2 \end{matrix} \end{pmatrix} = \begin{pmatrix} \boxed{1} & 0 \\ 0 & 1 \end{pmatrix} = I$

also $A^{-1} A = I$

$B =$ reflection across y-axis

$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



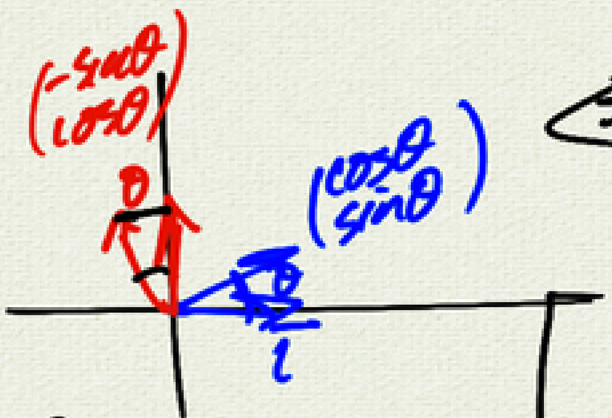
check: $B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$

$B^{-1} \stackrel{?}{=} B$ check $B \cdot B = I \Leftarrow B^2 = I$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$C =$ rotation by θ

$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$\Leftarrow \bar{u} \cdot \bar{v} = 0$ orthogonal

$|\bar{u}||\bar{v}| \cos\theta = 0$

check

$\boxed{\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}} \cdot \boxed{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}$

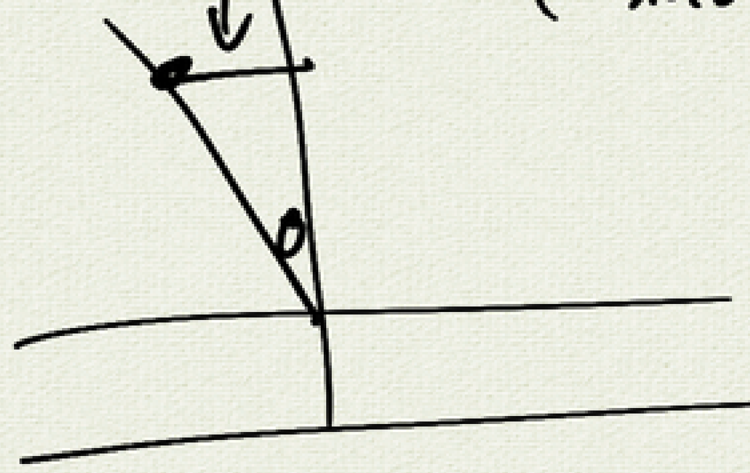
$= -\sin\theta\cos\theta + \cos\theta\sin\theta$

$= 0$

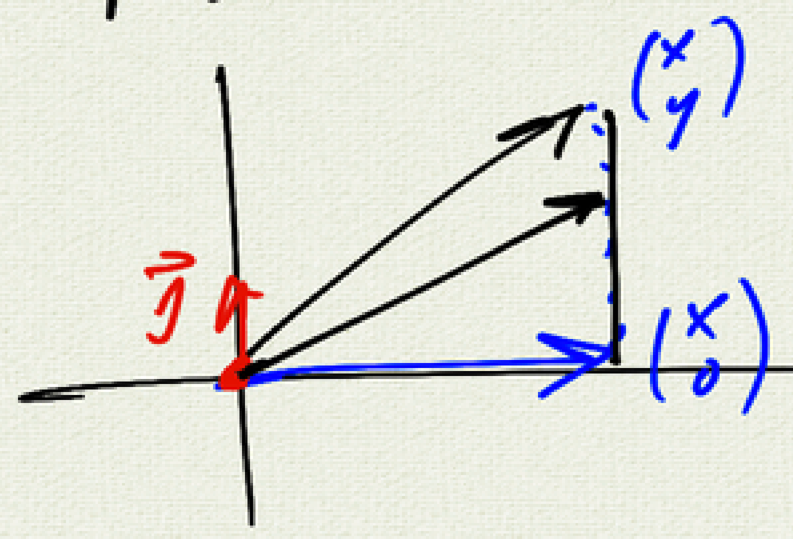
$C^{-1} =$ rotation by $-\theta$

$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

opposite $\underset{\text{x value}}{=}$ $= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Leftarrow$ exercise: $CC^{-1} = I$



D = projection on x-axis



$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

no inverse

(lost information)

not 1-1 map

---

explore:

$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{A} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \Leftarrow \quad ad-bc \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \boxed{(ad-bc)\, I}$$

$\Rightarrow A^{-1} = ?$

let $B = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$\Rightarrow AB = I \Leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity

$B = A^{-1}$

$$\boxed{A^{-1} = \frac{1}{\boxed{ad-bc}} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}}$$

A^{-1} exists $\Leftrightarrow \underbrace{ad-bc}_{\det A} \neq 0$

A^{-1} exists $\Leftrightarrow \det A \neq 0$

---

example $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \det A = |A| = ad-bc$

$= 4$

scale $\times 2$

$$\Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection

$\det B = ad-bc = -1$

$\boxed{\det B = -1} \rightarrow B^{-1}$ exists (invertible)

$$B^{-1} = \frac{1}{\boxed{ad-bc}} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$= \frac{1}{\boxed{-1}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{rotation}$$

$$\det C = \cos^2\theta + \sin^2\theta = 1$$

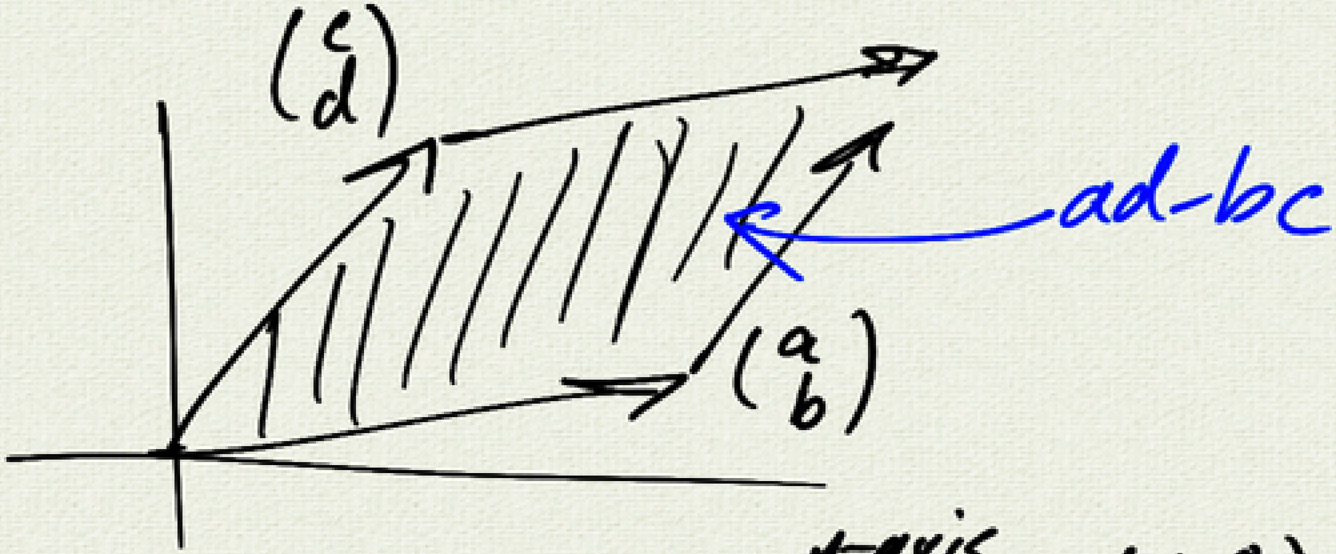
$$\Longrightarrow \quad C^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$= \frac{1}{1}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{projection}$$

$\det D = 0 \Longrightarrow D^{-1}$ does not exist

D is not invertible

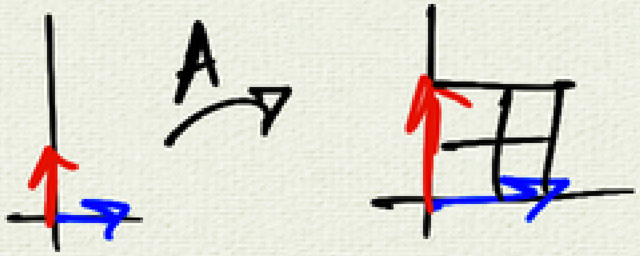


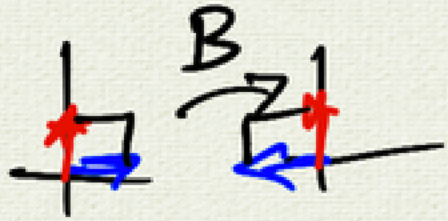
A scale ×2

$|A| = 4$

B y-axis reflection $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$|B| = -1$





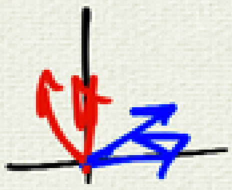
C rotation

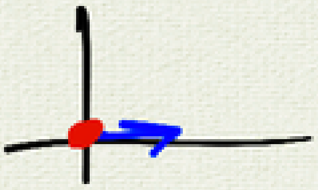
$|C| = 1$

D projection

$|D| = 0$

area = 0





example:

$$x + 2y = 2$$
$$3x + 5y = 3$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$ matrix equation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

if A^{-1} exists:

$$\underbrace{A^{-1}A}_{I}\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \qquad \det A = ad - bc$$
$$= 5 - 6$$
$$= -1$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$
$$= \frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$\rightarrow$ solution $\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

original: $x + 2y = 2$ $\leftarrow$ check ✓
$3x + 5y = 3$

example 2

$$x + 3z = 10$$
$$2x + y + 6z = 22$$
$$-2y + z = -1$$

matrix equation $\underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 0 & -2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \\ -1 \end{pmatrix}$

notation

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

A^{-1} exist?

$$\det A = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 0 & -2 & 1 \end{vmatrix} = +1 \begin{vmatrix} 1 & 6 \\ -2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 6 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}$$

negative

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= 13 - 0 + 3(-4)$$
$$= 1 \longleftarrow A^{-1} \text{ exists}$$

because $\det A \neq 0$

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 0 & -2 & 1 \end{pmatrix} \Longrightarrow A^{-1} = ?$$

$$\left(\underbrace{\begin{matrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 0 & -2 & 1 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) \longrightarrow (I \mid A^{-1})$$

$-2R_1 + R_2$

$$\left(\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right)$$

$2R_2 + R_3$

$$\left(\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 2 & 1 \end{matrix}\right)$$

$-3R_3 + R_1$

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 13 & -6 & -3 \\ -2 & 1 & 0 \\ -4 & 2 & 1 \end{matrix}}_{A^{-1}}\right)$$

solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 10 \\ 22 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 13 & -6 & -3 \\ -2 & 1 & 0 \\ -4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 22 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 130 - 132 + 3 \\ -20 + 22 + 0 \\ -40 + 44 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Longrightarrow \text{check in equations}$$

$x + 3z = 10$

$2x + y + 6z = 22$

$-2y + z = -1$