4.3 Polynomials

$x-2$ binomial (two terms)

$(x-2)(x-3) = x^2 - 5x + 6$ polynomial (degree 2, quadratic)

$$(x-1)(x-2)(x-3) = (x-1)(x^2-5x+6)$$
$$= x^3 - 5x^2 + 6x$$
$$\quad -x^2 + 5x - 6$$
$$= x^3 - 6x^2 + 11x - 6 \quad \text{(degree 3)}$$

$$p(x) = \underbrace{a_n x^n}_{\text{leading term}} + a_{n-1}x^{n-1} + \dots + a_1 x + \underbrace{a_0}_{\text{constant term}}$$

$\deg(p) = n$

$f(x) = x^3 - 6x^2 + 11x - 6 \Leftarrow f(0) = -6$ y-intercept given by constant term

$= (x-1)(x-2)(x-3) \Leftarrow$ see the zeros of $f(x)$ (1, 2, and 3)

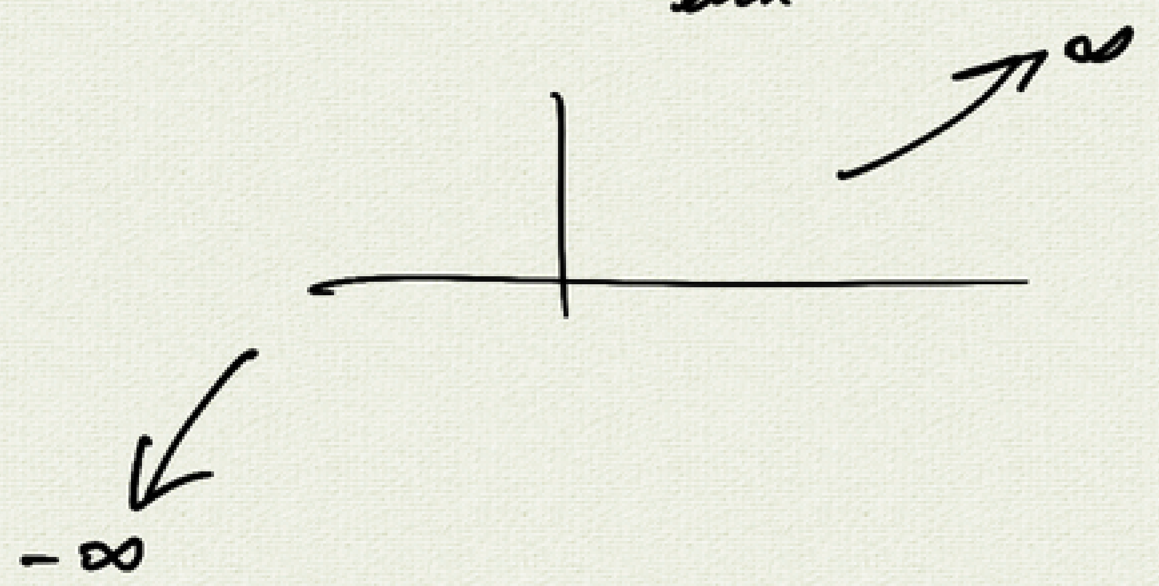
$f(1) = 0$
$f(2) = 0$
$f(3) = 0$

x	$f(x)$
1000	$1000^3 - 6 \cdot 1000^2 + \dots$
1,000,000	$1000000^3 - 6 \cdot 1000000^2$

leading term tells us about end behavior

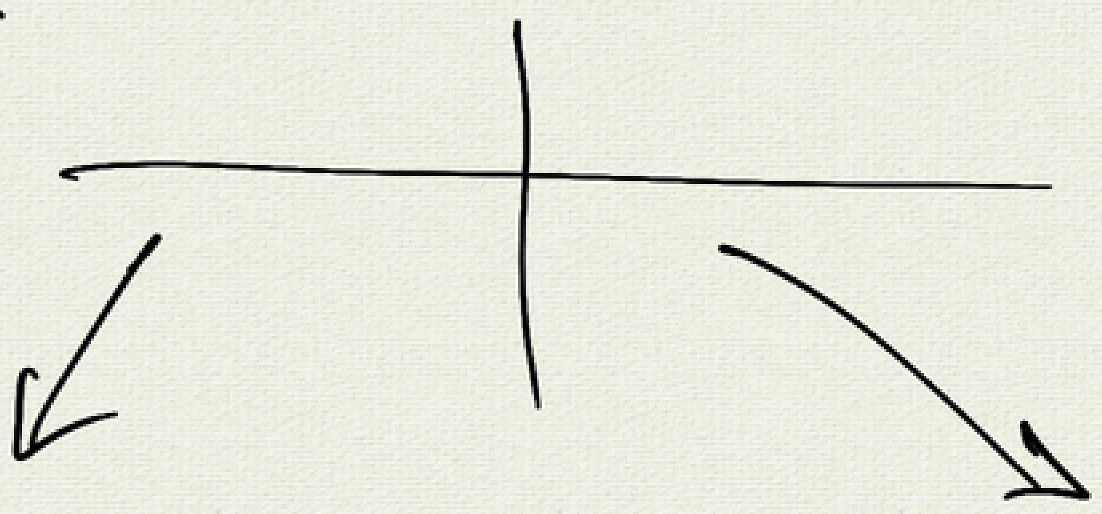
$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

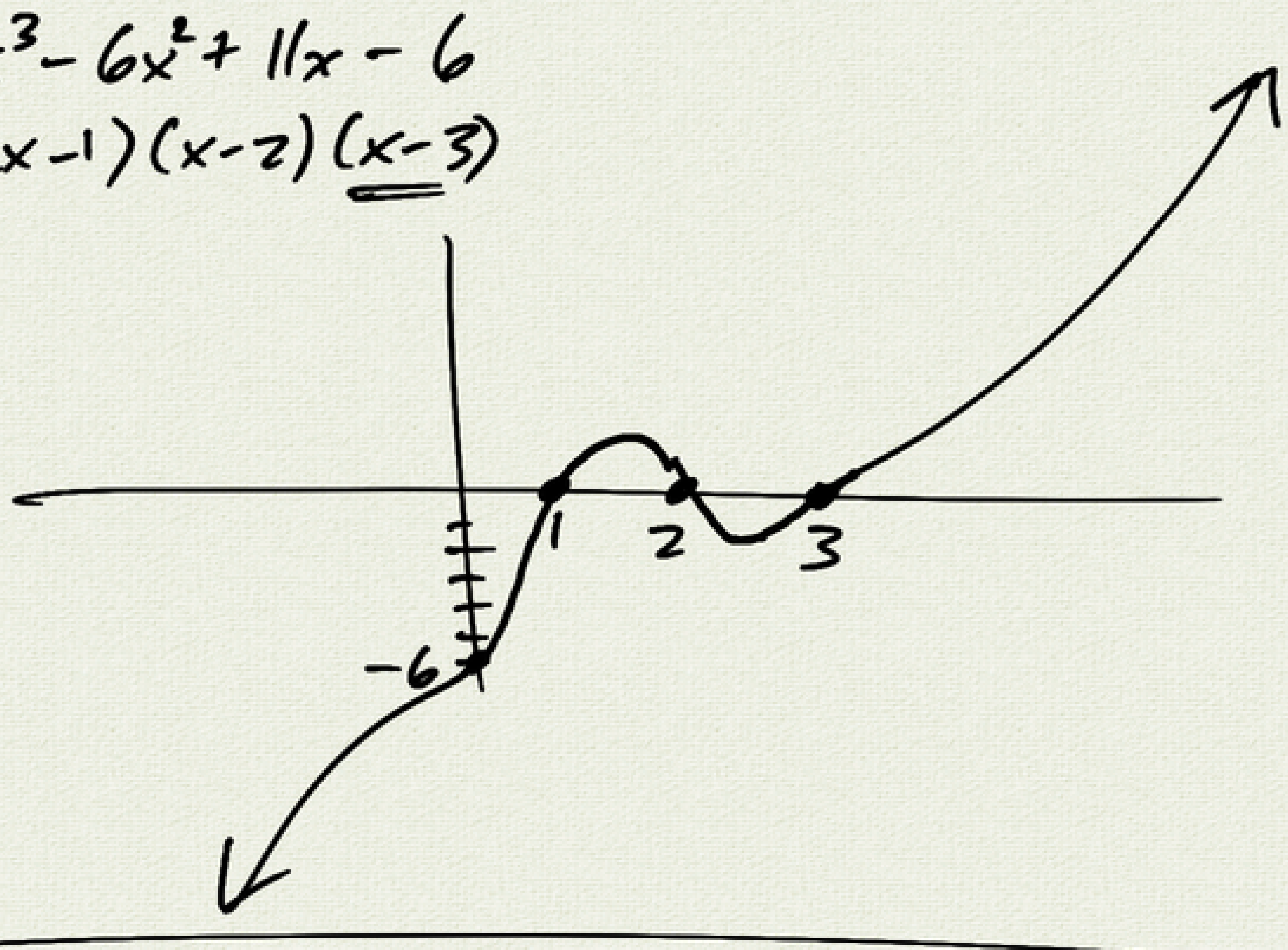


$g(x) = -4x^4 + \text{stuff}$

x	$g(x)$
-1000	$-4(1000)^4$
-1000000	$-4(1000000)^4$



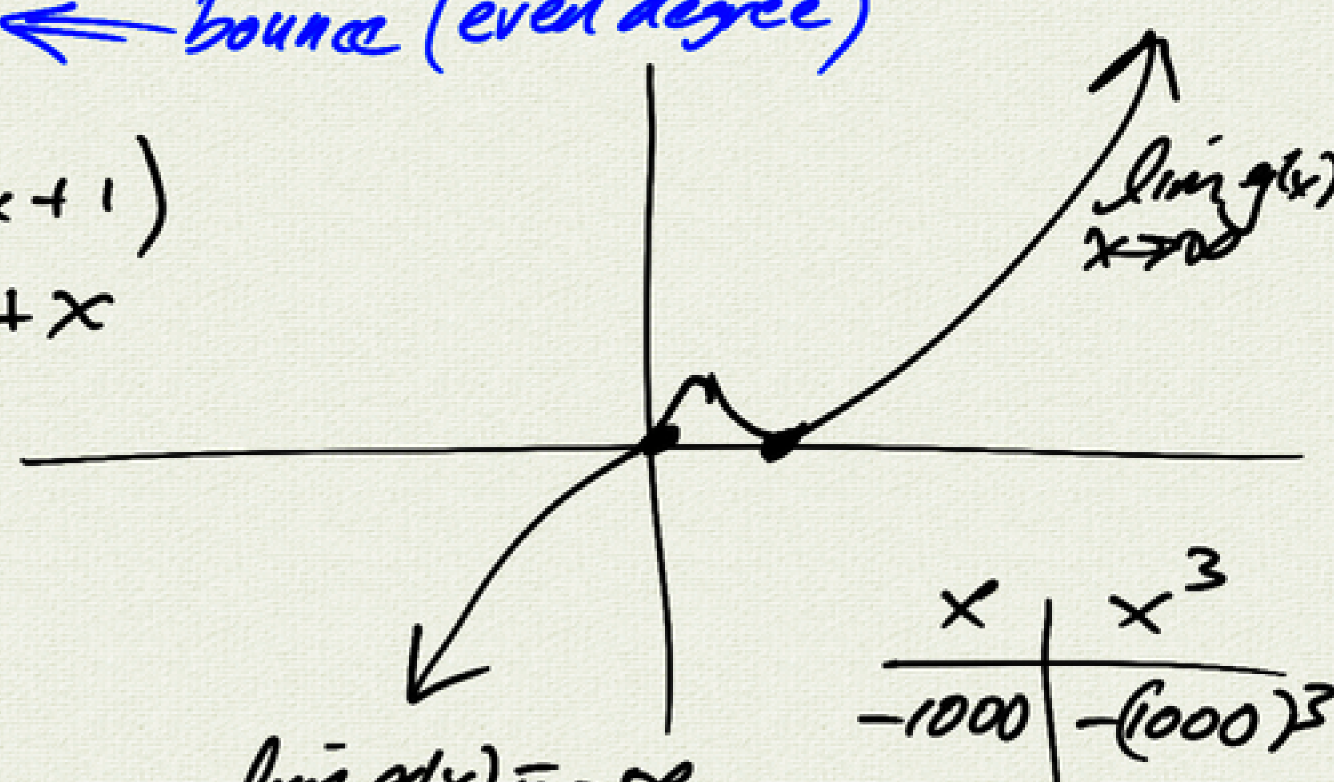
$f(x) = x^3 - 6x^2 + 11x - 6$
$= (x-1)(x-2)(x-3)$



$g(x) = x^1(x-1)^2 \Leftarrow$ bounce (even degree)

pierce (odd degree)

$= x(x^2 - 2x + 1)$
$= x^3 - 2x^2 + x$



$\lim_{x\to\infty} g(x) = \infty$

$\lim_{x\to-\infty} g(x) = -\infty$

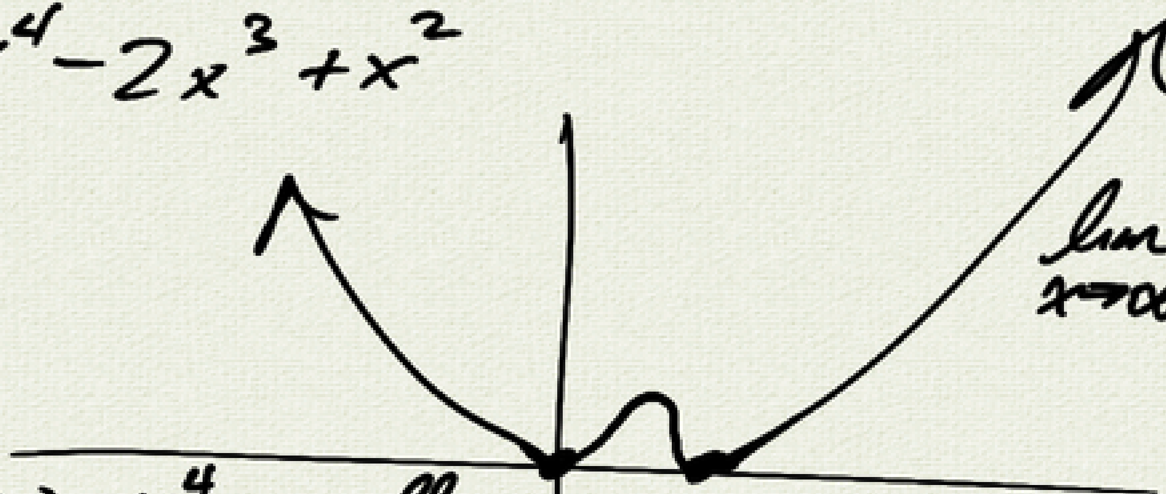
x	x^3
-1000	$-(1000)^3$

$$h(x) = x^2(x-1)^2$$

root multiplicity 2 (even $\Rightarrow$ bounce)

$$= x^2(x^2-2x+1)$$

$$= x^4 - 2x^3 + x^2$$



$$\lim_{x\to\infty} h(x) = +\infty$$

x	$h(x) = x^4 + \text{small}$
-1000	$(-1000)^4 = 1000^4$
-1000000	$(-1000000)^4 = 1000000^4$

$$\lim_{x\to-\infty} h(x) = +\infty$$

$$f(x) = (x-a)(\quad) \Rightarrow f(a) = 0$$

$x-a$ is a factor of f — a is a zero / root of f

$$x-a \mid f$$

"divides"

converse:

if a is a root of f, then $x-a$ is a factor

$$f(a)=0 \overset{?}{\Longrightarrow} x-a \mid f(x)$$

$$f(x) = x^3 - 6x^2 + 11x - 6$$
$$= \boxed{(x-1)(x-2)(x-3)}$$

hidden

division: divide $f(x)$ by $x-3$

$$\begin{array}{r l}
 & \boxed{x^2} - 3x + 2 \\
\boxed{x-3}) & x^3 - 6x^2 + 11x - 6 \\
 & -(x^3 - 3x^2) \\
 & \quad -3x^2 + 11x - 6 \\
 & \quad -3x^2 + 9x \\
 & \qquad 2x - 6 \\
 & \qquad 2x - 6 \\
 & \qquad \boxed{0}
\end{array}$$

remainder

$$\Rightarrow f(x) = (x^2 - 3x + 2)(x-3)$$

$$\frac{f(x)}{(x-3)} = \underline{x^2 - 3x + 2}$$

divisor

quotient

$g(x) = x^3 - 6x^2 + 11x - 5$ (add 1 to $f(x)$)

divide by $(x-1)$:

$$\begin{array}{rl}
 & x^2 - 5x + 6 \\
x-1 & \overline{)\, x^3 - 6x^2 + 11x - 5} \\
 & \underline{x^3 - x^2} \\
 & \quad -5x^2 + 11x - 5 \\
 & \quad \underline{-5x^2 + 5x} \\
 & \qquad\qquad 6x - 5 \\
 & \qquad\qquad \underline{6x - 6} \\
 & \qquad\qquad\quad \boxed{1} \text{ remainder}
\end{array}$$

$$g(x) = \underbrace{(x^2 - 5x + 6)}_{\text{quotient}}\underbrace{(x-1)}_{\text{divisor}} + \underbrace{1}_{\text{remainder}}$$

$$\frac{g(x)}{x-1} = x^2 - 5x + 6 + \boxed{\frac{1}{x-1}}$$

remainder as a fraction

$p(x)$ polynomial
$d(x)$ divisor

$\longrightarrow \quad p(x) = q(x)\,d(x) + r(x)$

(quotient)(divisor) remainder

$\deg(r) < \deg(d)$

degree 1

example: divisor $x-1$
remainder 1

degree 0

$p(x)$ polynomial

if $x-a \mid p(x)$, then $p(a)=0$

$x-a$ factor $\Rightarrow$ a root

$5x^0$ degree 0

x^1+5x^0 degree 1

$x^2+2x^1+5x^0$ degree 2

try to divide by $x-a$:

$$\underset{\text{quotient}}{p(x)=q(x)}\,\underset{\text{divisor}}{(x-a)} + r(x)$$

$\deg(r) < \deg(d) = 1$

$$\boxed{p(x) = q(x)(x-a) + r}$$

constant

$p(a) = ?$

$p(a) = r$

Remainder Theorem

The remainder when you divide by $x-a$ is $p(a)$.

$11 \div 3 = 3R2$

integers

$p = q \cdot d + r$

quotient $\cdot$ divisor + remainder

$11 = 3 \cdot 3 + 2$

$r < d$

Suppose $p(a)=0$ (a is a root of p)

then divide by $x-a$:

$$p(x) = q(x)(x-a) + r$$

$$p(a) = r$$

$$p(a) = 0 \Rightarrow r = 0$$

$$\Rightarrow p(x) = q(x)(x-a)$$

$\Rightarrow$ $x-a$ is a factor of p

Summary

$p(x)$ polynomial

$x-a$ is a factor of p $\Longleftrightarrow$ a is a root of p

$x-a \mid p$ $\qquad$ $p(a)=0$

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$= (x-1)(x-2)(x-3)$$

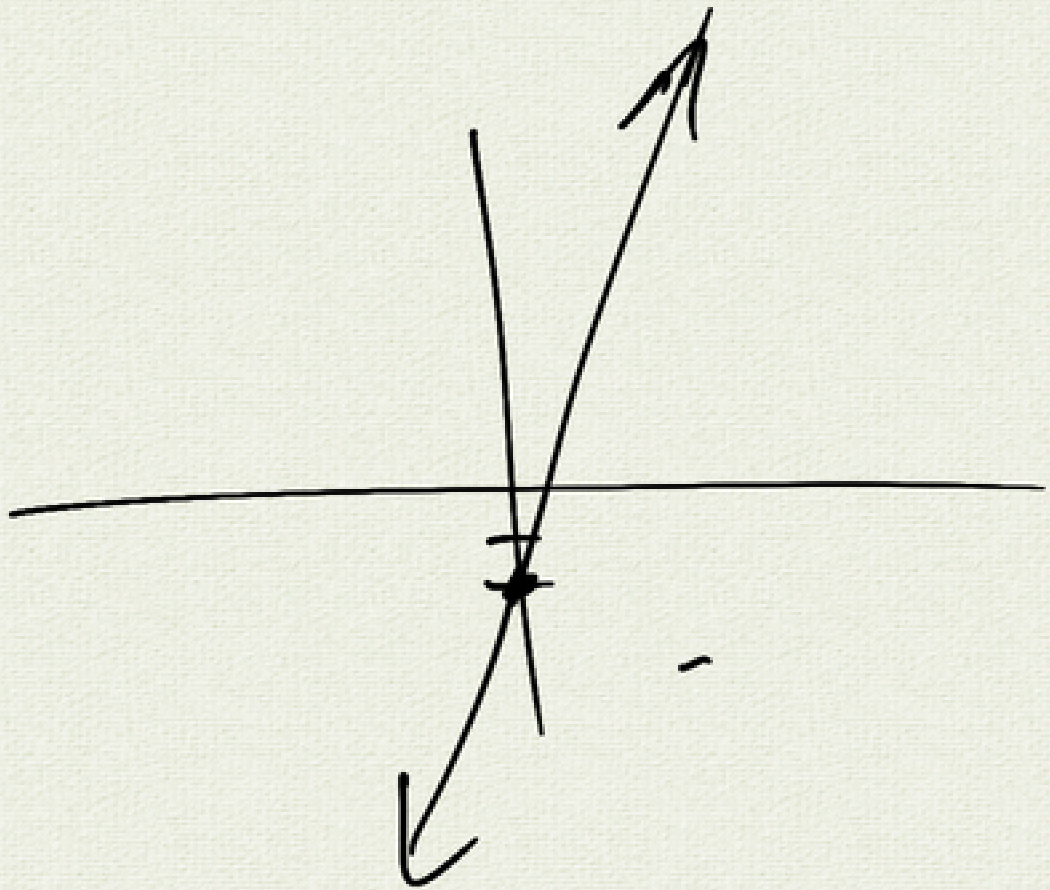
factors:		roots	
$x-1$	$\longleftrightarrow$	1	$f(1)=0$
$x-2$	$\longleftrightarrow$	2	$f(2)=0$
$x-3$	$\longleftrightarrow$	3	$f(3)=0$

$\boxed{5.2}$

① $f(x) = 6x - 2$

$y = 6x - 2$

inverse:
① swap x/y
② solve for y



$x = 6y - 2$

$x + 2 = 6y$

$y = \frac{1}{6}(x+2)$

$$\boxed{f^{-1}(x) = \frac{1}{6}(x+2)}$$