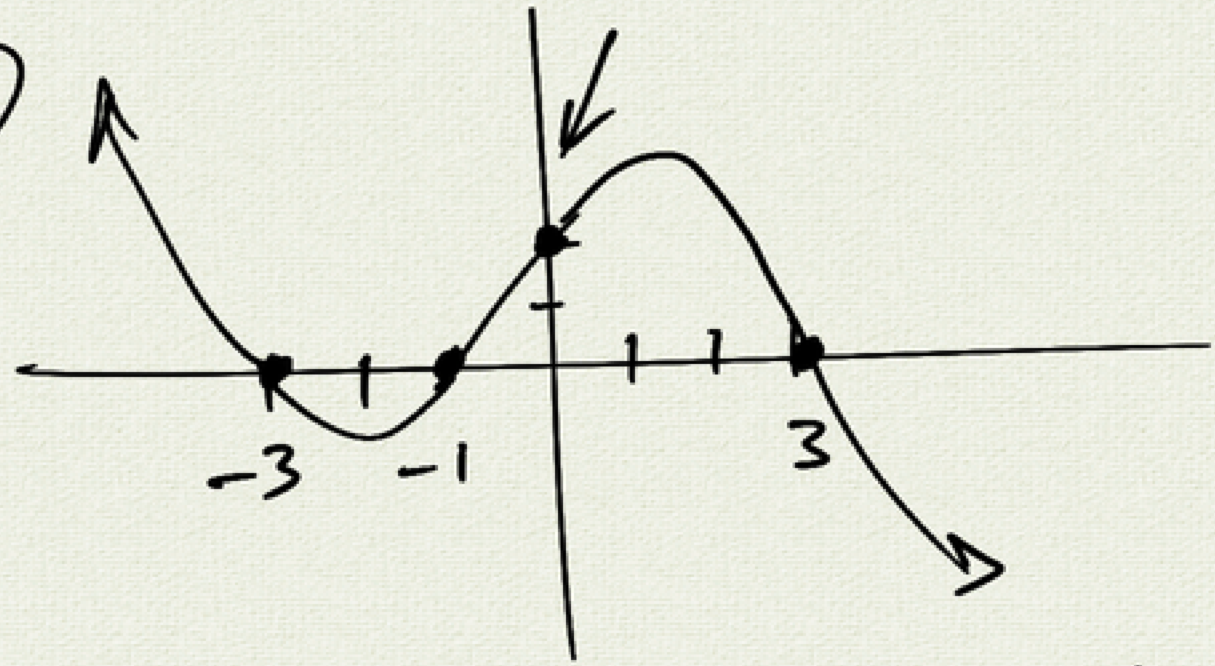
$\boxed{3.4}$

49



$$\Longrightarrow \quad p(x) = k(x+3)(x+1)(x-3)$$

$$p(0) \; (= a_0) = k(-9) = 2$$

$$\Longrightarrow \quad k = -\frac{2}{9}$$

$$p(x) = -\frac{2}{9}(x+3)(x+1)(x-3)$$

4.5 Fundamental Theorem of Algebra

polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$x-a$ factor $\Longleftrightarrow$ a is a zero (root)

$$x-a \mid p(x) \quad \Longleftrightarrow \quad p(a) = 0$$

$\deg(p) = n \Longrightarrow$ at most n factors $\qquad (x-1)(x-2)$

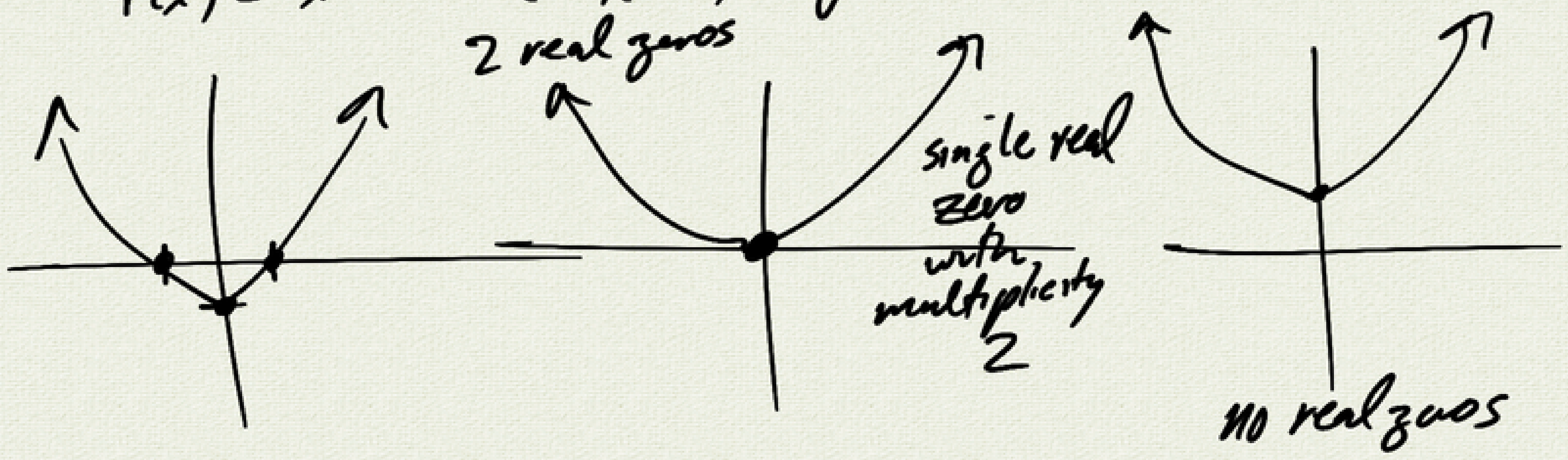
at most n zeros

Fundamental Theorem of Algebra: $p(x)$ has exactly n zeros

(but some may be complex)

Simple examples

$f(x) = x^2 - 1 = (x+1)(x-1)$ $\qquad g(x) = x^2$ $\qquad h(x) = x^2+1$



$h(x) = x^2+1$

$h(x)=0 \Longrightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{0 \pm \sqrt{0-4}}{2} = \pm\dfrac{\sqrt{-4}}{2} = \pm\dfrac{\sqrt{4}\sqrt{-1}}{2}$

$= \pm\sqrt{-1}$ $\leftarrow i$

$= \pm i$

complex number:

$a + bi$ $\qquad i^2 = -1$

real $\quad$ real

$z_1 = 2+3i$
$z_2 = 4+5i$ $\Big\}$ $z_1 + z_2 = 6 + 8i$

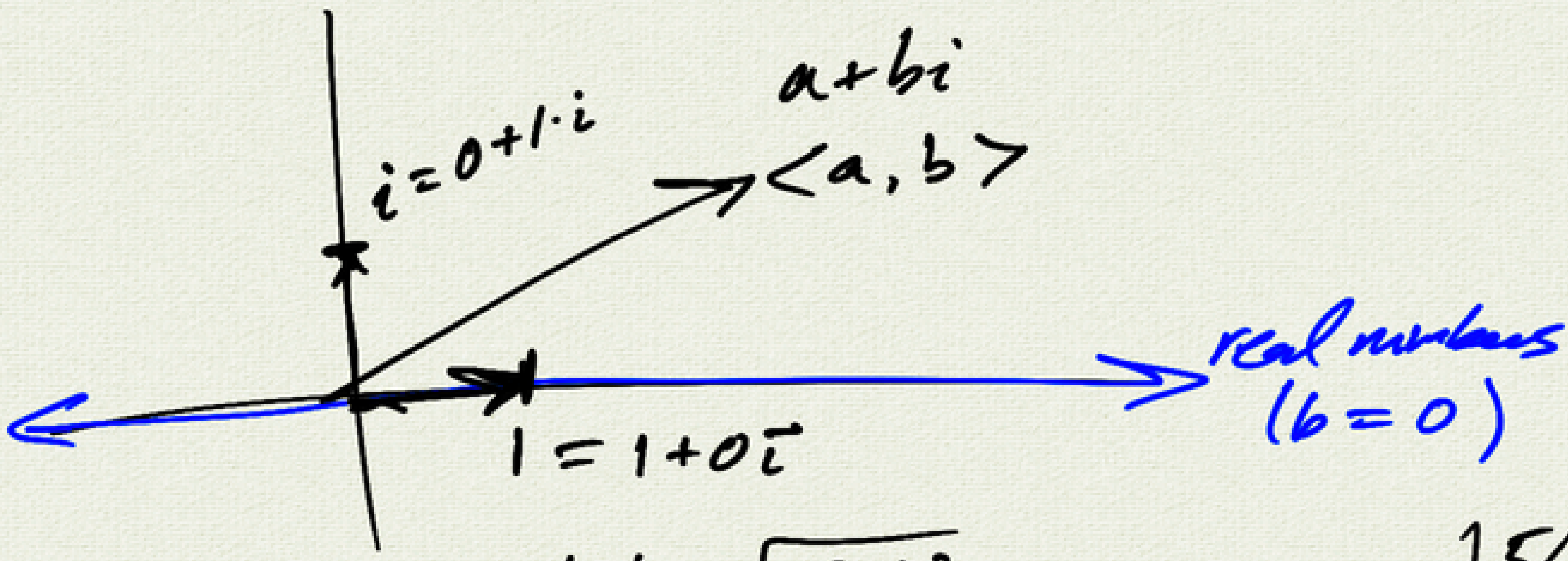
real part (x component) $\quad$ imaginary part (y-component)

$5z_1 = 5(2+3i)$
$= 10 + 15i$

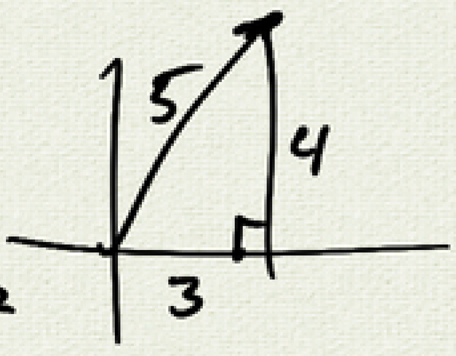
$z_1 z_2 = (2+3i)(4+5i)$

multiply $\quad = 8 + 10i + 12i + \underbrace{15i^2}_{-15}$

$= -7 + 22i$



$z = a+bi \Longrightarrow |z| = \sqrt{a^2+b^2}$



$(3+4i)(3-4i) = 9 - (4i)^2 = 9 - 16i^2$
$= 9 + 16$
$= 25$

$z = a+bi \Longrightarrow \bar{z} = a - bi$ "z-conjugate"

$z\bar{z} = |z|^2$

$h(x) = x^2+1$ ⟸ irreducible quadratic

$= (x+i)(x-i)$ ⟵ check

$(x+i)(x-i) = x^2 - i^2$

$= x^2+1$

Zeros: $x^2+1=0$

$x^2 = -1$

$x = \pm\sqrt{-1}$

$x = \pm i$

Summary: $p(x)$ polynomial, degree n

$x-a$ factor $\Longleftrightarrow$ a is a zero

Fundamental Thm: $p(x)$ has n linear complex factors

n complex zeros

Also: complex roots occur in conjugate pairs

(quadratic formula: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ ⟵ conjugates) example:

$q(x) = x^4 + 2x^2 + 8x + 5$

factor completely / find all zeros

potential rational zeros: $\pm 1, \pm 5$

$q(1) = 1 + 2 + 8 + 5 \neq 0$

$q(-1) = 1 + 2 - 8 + 5 = 0$

$$
\begin{array}{r|rrrrr}
-1 & 1 & 0 & 2 & 8 & 5 \\
 & & -1 & 1 & -3 & -5 \\
\hline
 & 1 & -1 & 3 & 5 & \boxed{0}
\end{array}
$$

$x^3 - x^2 + 3x + 5$

$q(x) = (x+1)(x^3 - x^2 + 3x + 5)$

$q(-1) = (0)\,\underbrace{(-1 - 1 - 3 + 5)}_{0}$

$$
\begin{array}{r|rrrr}
-1 & 1 & -1 & 3 & 5 \\
 & & -1 & 2 & -5 \\
\hline
 & 1 & -2 & 5 & \boxed{0}
\end{array}
$$

$\Longrightarrow \boxed{q(x) = (x+1)^2(x^2 - 2x + 5)}$ factored over $\mathbb{R}$

$x^2 - 2x + 5 = 0 \Rightarrow x = \dfrac{2 \pm \sqrt{4-20}}{2}$

$= 1 \pm \dfrac{\sqrt{-16}}{2}$

$= 1 \pm 2i$

$\sqrt{-16} = \sqrt{-1}\sqrt{16}$
$= 4i$

$\boxed{q(x) = (x+1)^2(x - (1+2i))(x - (1-2i))}$ factored over $\mathbb{C}$ complex numbers

n linear factors
(degree 1)
$x - a$

$\underbrace{(x-(-1))(x-(-1))(x \quad)(\quad)}$
n factors that look like $x - a$

$i^3 = i(i^2) = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

---

example

$p(x) = x^4 - 6x^3 + x^2 + 54x - 90$

factor completely (over $\mathbb{R}$ and $\mathbb{C}$) / find all zeros

little bird: $3+i$ is a root

$$\begin{array}{r|rrrrr}
3+i & 1 & -6 & 1 & 54 & -90 \\
 & & 3+i & -10 & -27-9i & 90 \\
\hline
3-i & 1 & -3+i & -9 & 27-9i & \boxed{0} \\
 & & 3-i & 0 & -27+9i & \\
\hline
 & 1 & 0 & -9 & \boxed{0} &
\end{array}$$

$(3+i)(-3+i)$
$= -9 - 1$
$= -10$

$(3+i)(27-9i)$
$= 81 + 9$
$= 90$

$\Rightarrow p(x) = (x-(3+i))(x-(3-i))(x^2-9)$

$$\boxed{p(x) = \underbrace{(x-(3+i))(x-(3-i))}(x+3)(x-3)}$$ factored over $\mathbb{C}$

2 real roots ± 3

$(x-(3+i))(x-(3-i))$
$= x^2 - (3+i)x - (3-i)x + (3+i)(3-i)$
$= x^2 - 6x + 10$

$9+1$

$$\boxed{p(x) = (x^2-6x+10)(x+3)(x-3)}$$ factored over $\mathbb{R}$

irreducible quadratic