4.6 Rational Functions

$$\frac{p(x)}{q(x)} \leftarrow \text{polynomial}$$

rational #'s $\mathbb{Q}$

$$\frac{p}{q} \leftarrow \text{integer } \mathbb{Z}$$

$0, \pm 1, \pm 2, \pm 3, \ldots$

$\frac{1}{2} \in \mathbb{Q}$

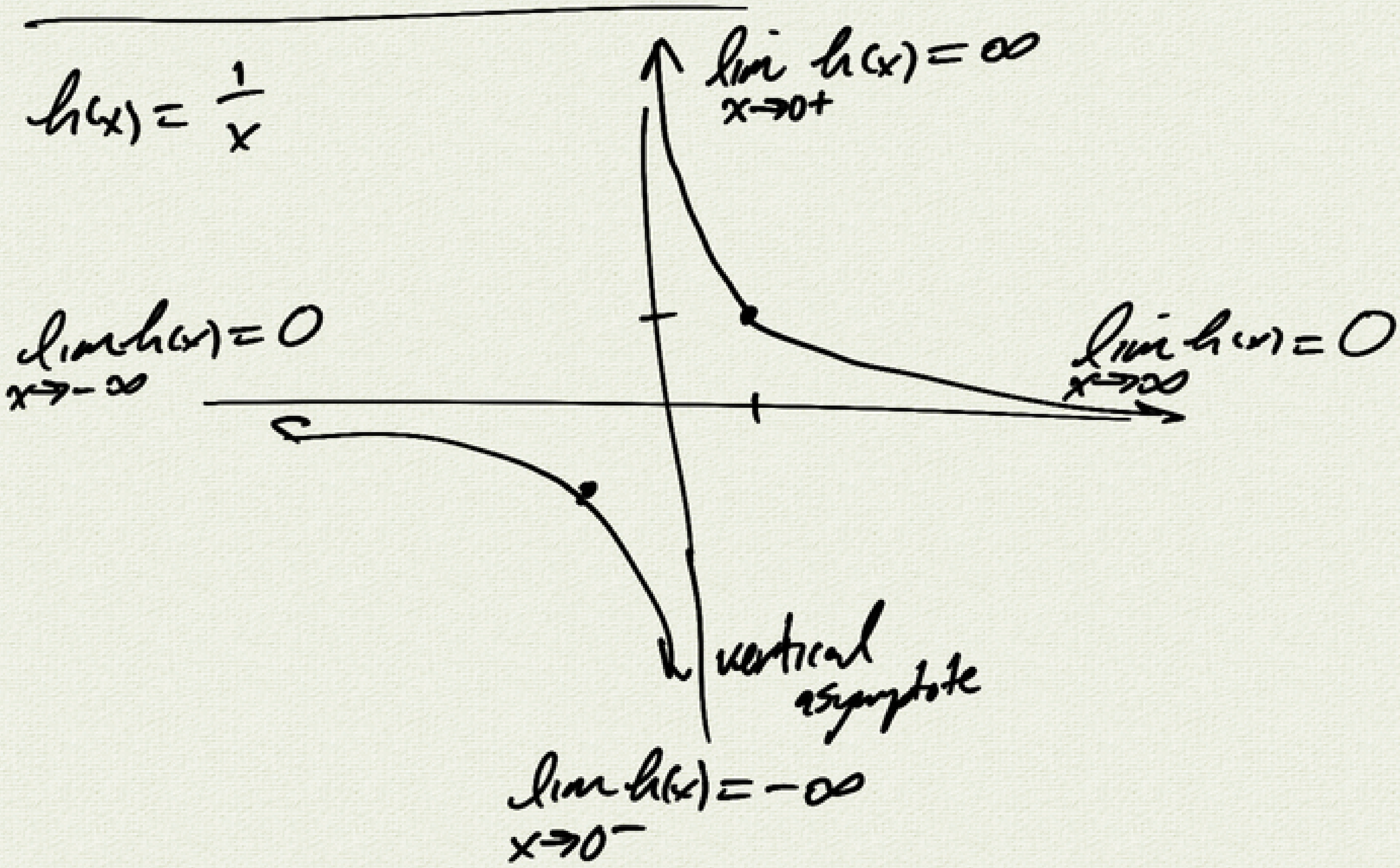
$f(x) = 1$

$g(x) = \frac{x^3 + 2x^2 + x + 1}{1}$ any polynomial is a rational function

$1 = 1 \cdot x^0$

$\frac{x}{1} = x$

$h(x) = \frac{1}{x}$



$\frac{1}{\text{small}} = \text{big}$ $\frac{1}{\text{big}} = \text{small}$

polynomial: $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

$$k(x) = \frac{x-2}{x-1}$$

end behavior? $x \to \infty$

$$\lim_{x\to\infty} k(x) = 1$$

x	$k(x)$
1000000	$\frac{1000000-2}{1000000-1} \approx 1$

hypothetical:

$$\frac{2x-2}{x-1} \longrightarrow 2$$

as $x \to \infty$

$$\frac{3x-1}{5x-7} \longrightarrow \frac{3}{5}$$

as $x \to \infty$

$$\frac{3x-1}{5x^2-7} \longrightarrow 0$$

x^2 grows faster than x

$$\frac{3x^2-1}{5x-7} \longrightarrow \infty$$

x^2 grows faster than x

Summary:

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \dots + a_0}{b_n x^n + \dots + b_0} \qquad \deg(p) = m, \quad \deg(q) = n$$

end behavior: look leading terms

$\underline{m < n}$ $\quad \lim_{x\to\infty} r(x) = 0$

$\underline{m = n}$ $\quad \lim_{x\to\infty} r(x) = \frac{a_m}{b_n}$

$\underline{m > n}$ $\quad \lim_{x\to\infty} r(x) = (\pm)\infty$

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example: $\boxed{k(x) = \frac{x-2}{x-1}}$ $\qquad \frac{a_1 x - a_0}{b_1 x - b_0} \quad m=1, \; n=1$

$\Rightarrow \lim_{x\to\infty} k(x) = 1$

$h(x) = \frac{1}{x} \qquad \frac{a_0}{b_1 x + b_0} \qquad m = 0, \; n = 1$

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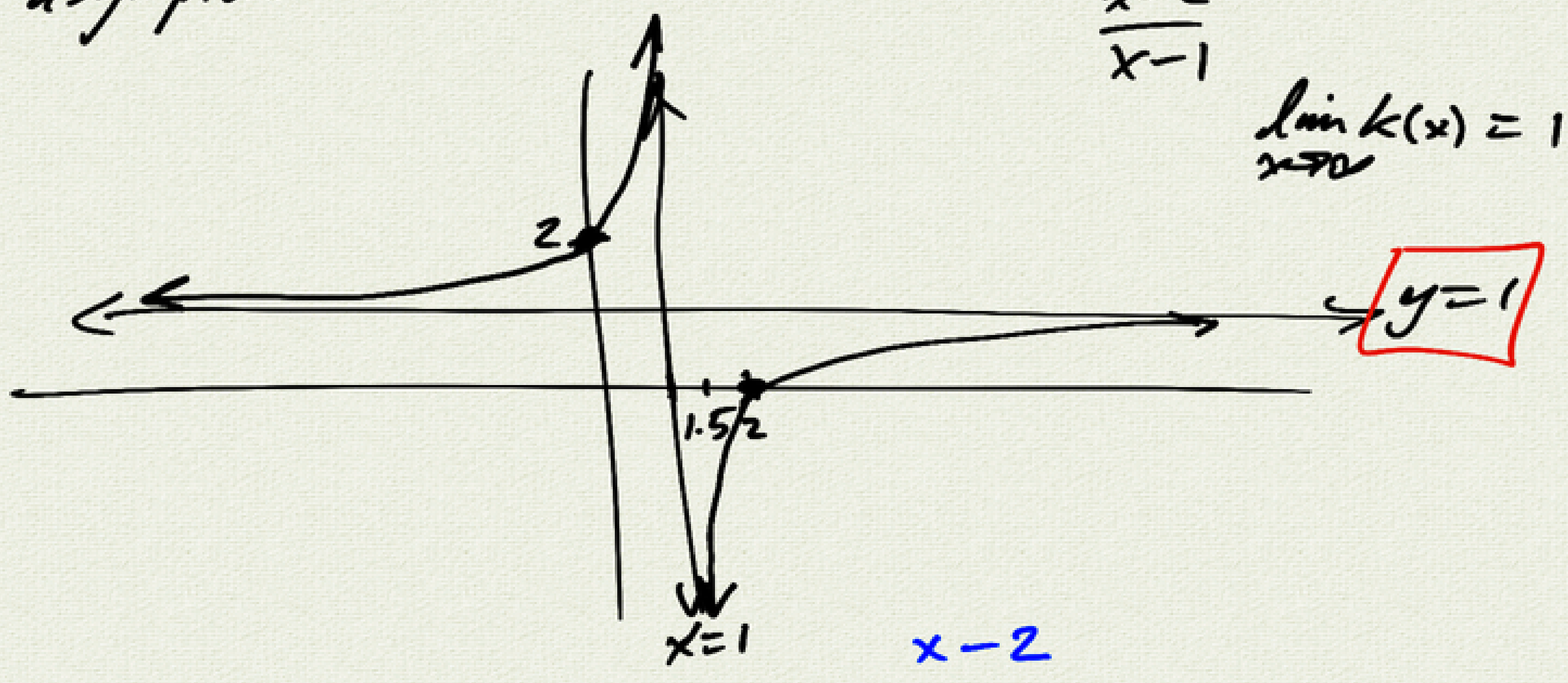
asymptote at $x=1$



$k(x) = \frac{x-2}{x-1}$

$x-1$ $\qquad$ $x-2$

$$\begin{array}{r|rr} 1 & 1 & -2 \\ & & 1 \\ \hline & 1 & \boxed{-1} \end{array}$$

$x-a \leftrightarrow a$

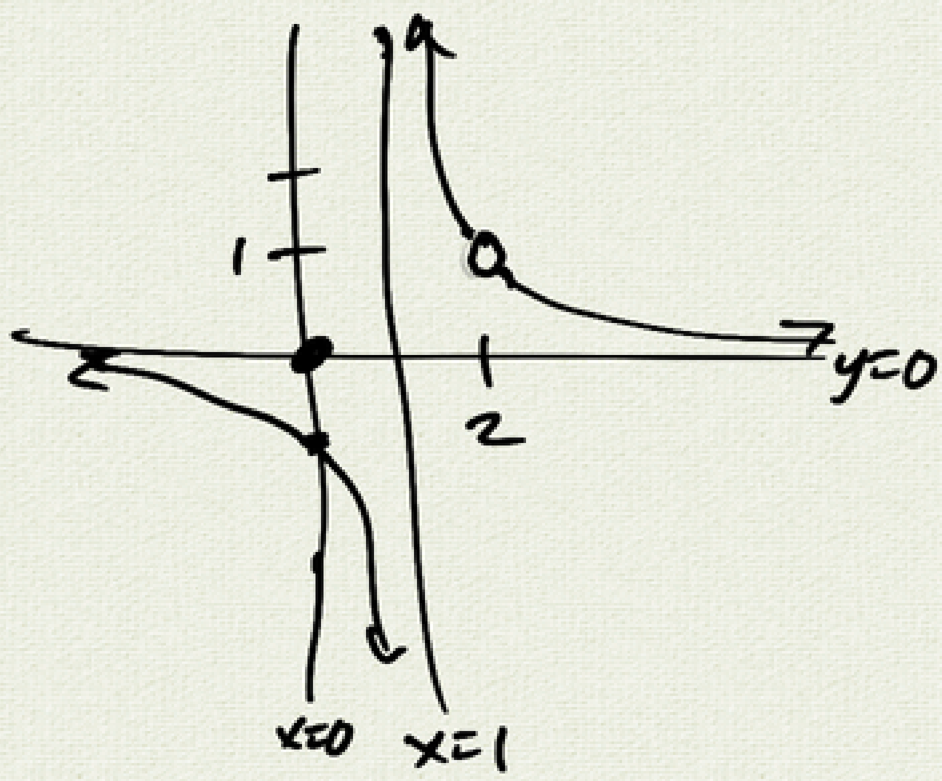
factor $\quad$ zero

$$k(x) = 1 - \frac{1}{x-1} \approx \frac{1}{x}$$

shift up 1 $\qquad$ flip $\qquad$ shift right 1

$$\ell(x) = \frac{x-2}{(x-1)(x-2)}$$

$$= \begin{cases} \frac{1}{x-1} & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$$



discontinuous at $x=2$

<u>removable</u> $\ell(2)$ should be 1

$$m(x) = \frac{x-3}{(x-2)(x+3)} \left(= \frac{x-3}{x^2+x-6} \right)$$

asymptotes at $\boxed{x=2,\ x=-3}$

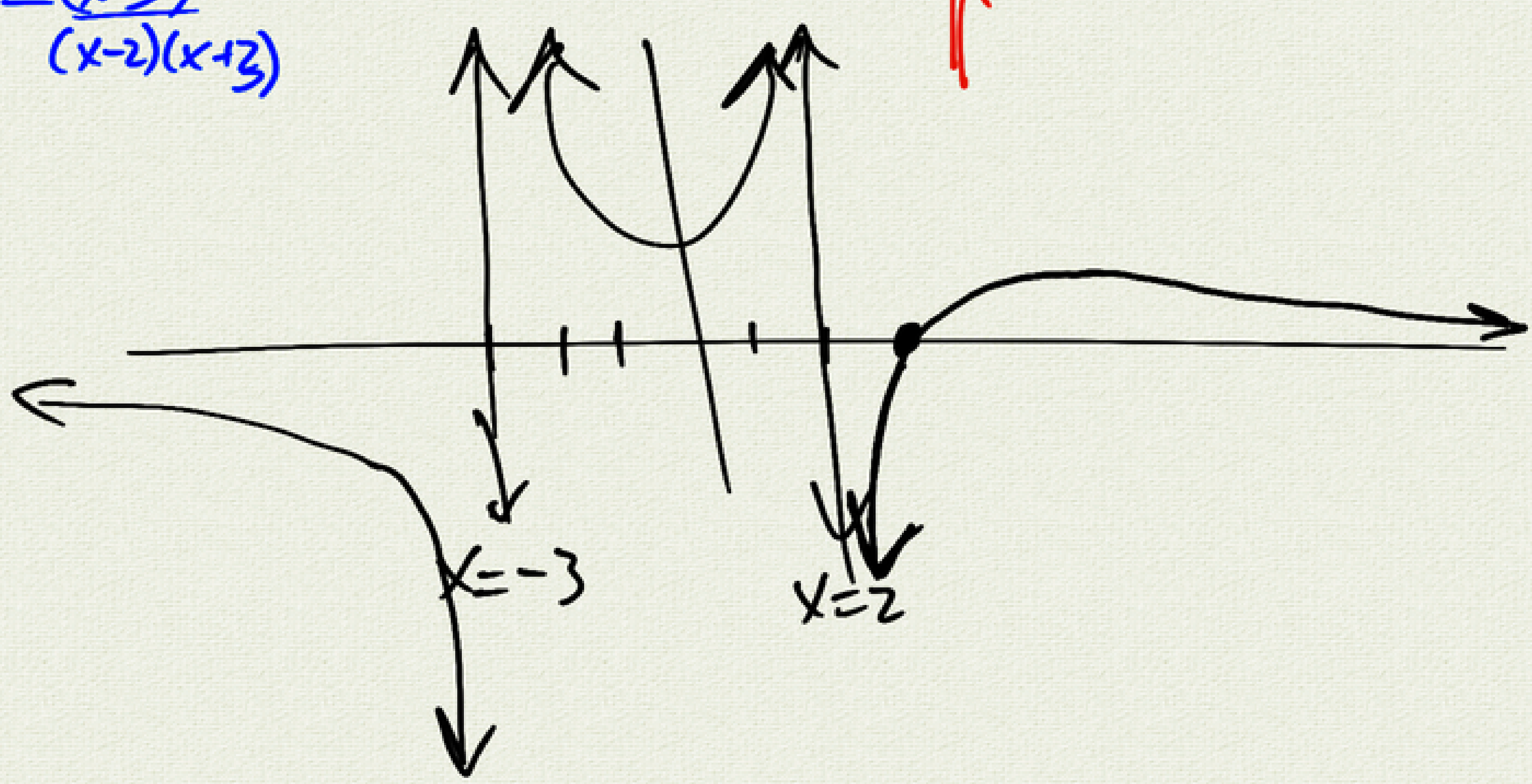
Zero at $\boxed{x=3}$

$m(3) = \frac{0}{\text{something}} = 0$

<u>end behavior:</u> $\lim_{x \to \infty} m(x) = 0$

		-3	$(-3,2)$	2		3	
$x+3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$m(x)$	$-$	X	$+$	X	$-$	0	$+$

$= \frac{(x-3)}{(x-2)(x+3)}$



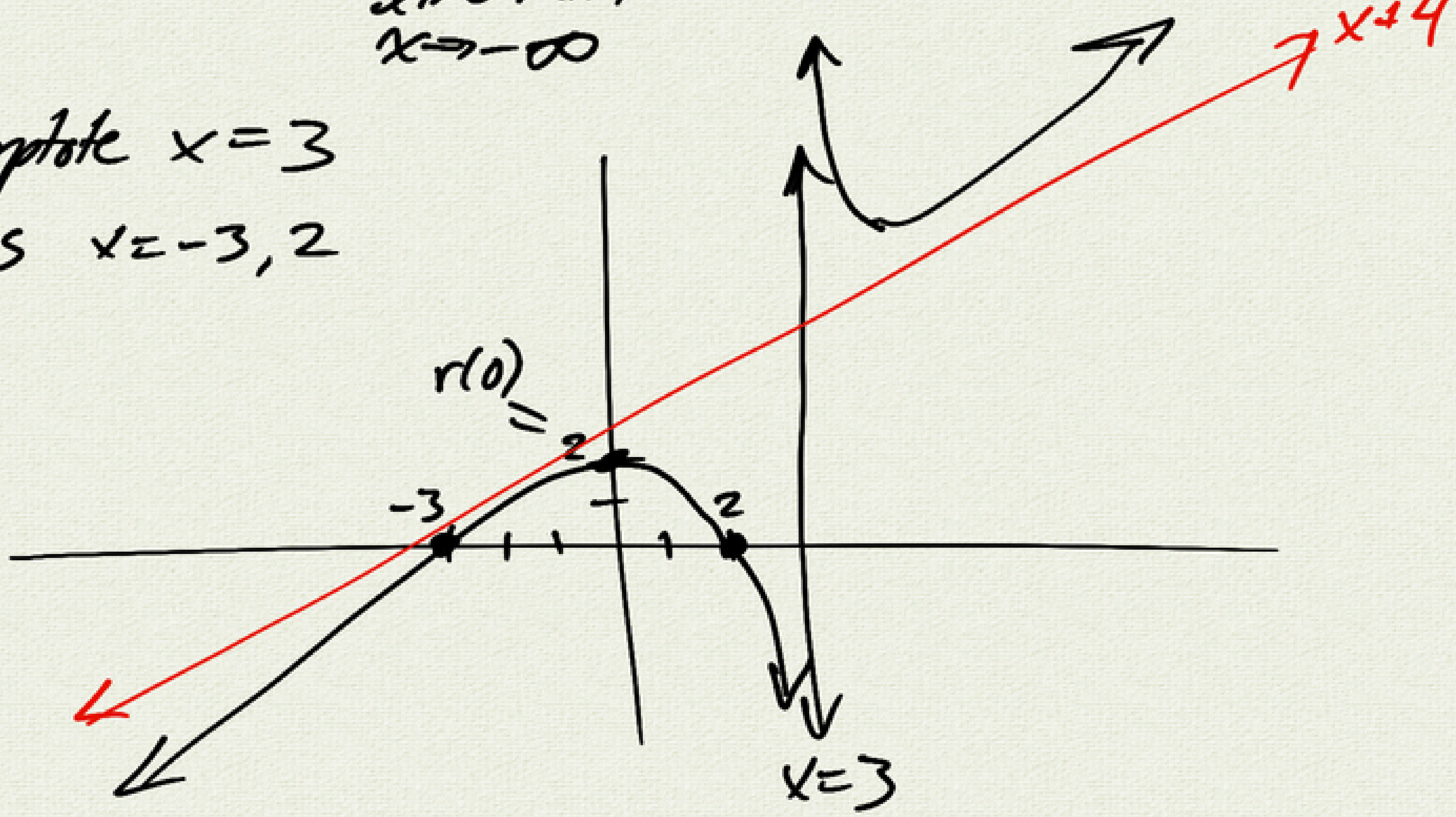
$$r(x) = \frac{(x+3)(x-2)}{x-3}$$

end behavior: $\lim_{x\to\infty} r(x) = \infty$

$\lim_{x\to-\infty} r(x) = -\infty$

asymptote $x=3$

zeros $x=-3, 2$



$$r(x) = \frac{(x+3)(x-2)}{x-3} = \frac{x^2+x-6}{x-3}$$

divide:

$$\begin{array}{r|rrr} 3 & 1 & 1 & -6 \\ & & 3 & 12 \\ \hline & 1 & 4 & \boxed{6} \end{array}$$

$$\Rightarrow r(x) = x+4+\boxed{\frac{6}{x-3}} \to 0 \text{ as } x\to\infty$$

$r(x) \approx x+4$ as $x\to\infty$

slant asymptote