② $\{a_k\} = \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots$

$r = -\frac{1}{2}$

geometric

$a_1 = \frac{1}{2}$

recursive: $a_1 = \frac{1}{2}$

$a_{n+1} = a_n\left(-\frac{1}{2}\right)$

$\left(a_{n+1} = -\frac{1}{2}a_n\right)$

explicit: $a_n = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1}$

$a_k = \frac{1}{2}\left(-\frac{1}{2}\right)^{k-1}$

$a_n = -\frac{1}{2}a_{n-1}$

$a_n = r\,a_{n-1}$ recursive

$a_n = a_1 \cdot r^{n-1}$ explicit

$$S_n = a_{\textcircled{1}} + a_2 + \dots + a_{\textcircled{n}}$$

$$= \sum_{k=1}^{n} a_k$$

end → n

start → $k=1$

$\boxed{1}, \boxed{\frac{1}{2}}, \frac{1}{4}, \frac{1}{8}, \dots$

$b_0 \quad b_1 \quad b_2 \quad b_3 \dots$

$$b_k = \left(\frac{1}{2}\right)^k$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \sum_{k=0}^{3} b_k$$

$$\sum_{k=3}^{5} b_k = b_3 + b_4 + b_5 = \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$$

$a_1 = \frac{1}{2}$

$r = -1/2$

finite sum of geometric

$$= \frac{\frac{1}{2}\left(1-\left(-\frac{1}{2}\right)^n\right)}{\boxed{1-\left(-\frac{1}{2}\right)}} \quad 3/2$$

$$= \frac{1}{3}\Big(1 - \underbrace{\left(-\tfrac{1}{2}\right)^n}_{\to 0 \text{ as } n\to\infty}\Big) \Longrightarrow \lim_{n\to\infty} S_n = \frac{1}{3}$$

$$S_\infty = \frac{a_1}{1-r} = \frac{1/2}{1-(-1/2)} = \frac{1}{3}$$

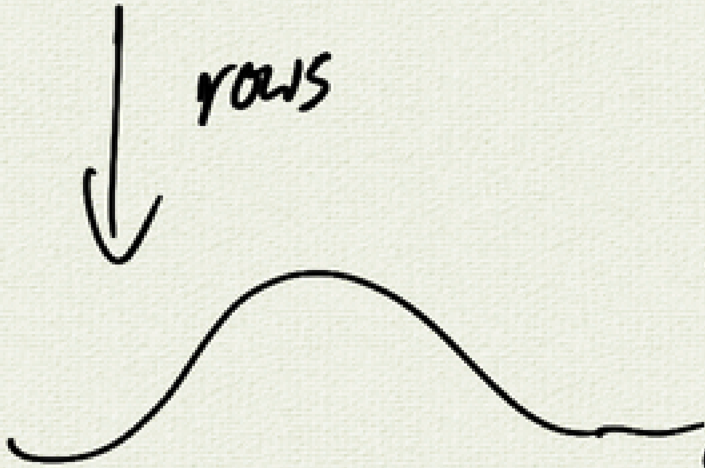
infinite sum

"infinite series"

$$1 + x + x^2 + \dots = \frac{1}{1-x} \qquad |x| < 1$$

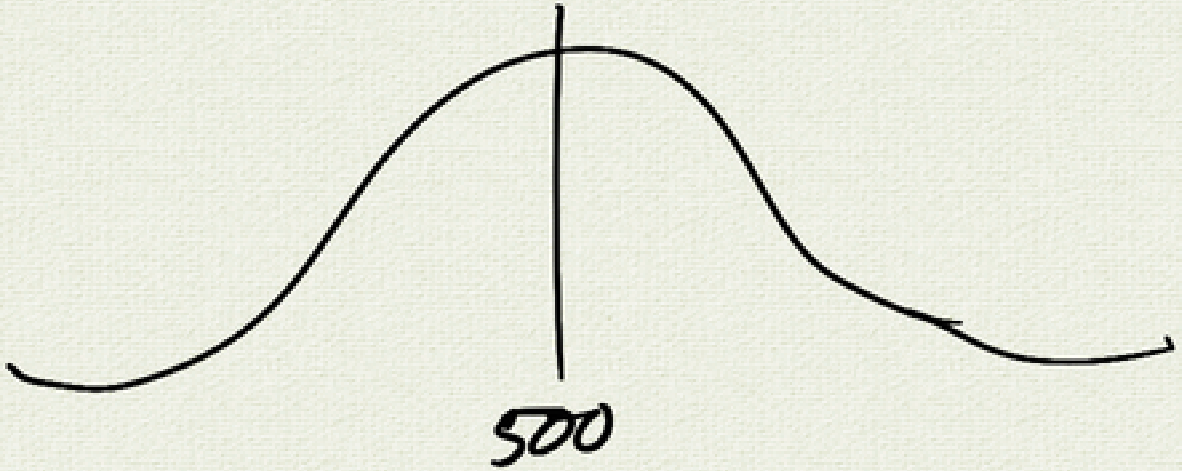
$$1 + \frac{1}{2} + \frac{1}{4} + \dots = 2 \quad \left(= \frac{1}{1-1/2}\right)$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1



bell curve
normal distribution
Gaussian

Central Limit Theorem



rational #'s

$$\frac{p}{q}$$

integer

decimal ⇒ finite or repeating

$\frac{1}{2} = .5$

$\frac{1}{3} = .333\ldots = .\overline{3}$

irrational:

non-repeating decimal

$\pi = 3.14159\ldots$

$e = 2.71828\ldots$

$$5.\underline{\underline{123}}\,\underline{\underline{123}}\,123\ldots = 5.\overline{123}$$

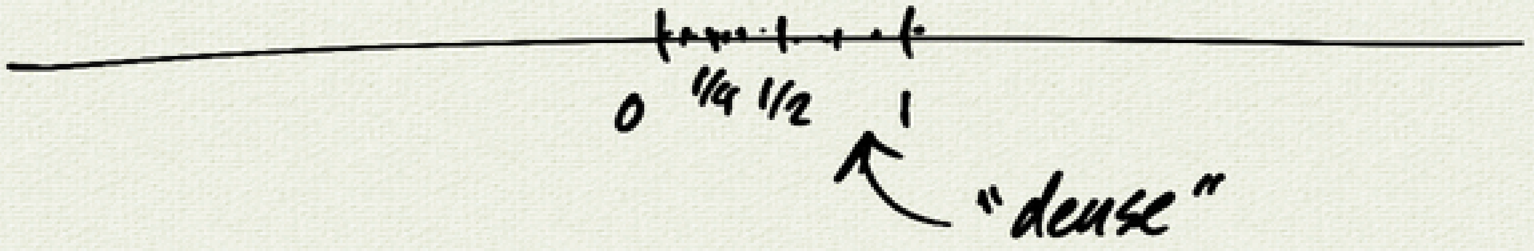
$$= 5 + \underbrace{\boxed{\frac{123}{1000}} + \boxed{\frac{123}{(1000)^2}} + \frac{123}{(1000)^3} + \ldots}_{\text{geometric}} \qquad r = \frac{1}{1000}$$

$$S_\infty = \frac{a_1}{1-r} = \frac{123/1000}{1 - 1/1000}$$

$$= \frac{123/1000}{999/1000}$$

$$= 123/999$$

$$= 5\frac{123}{999}$$



Cantor - set theory

measure size of set

$\{A, B, C\} \longleftrightarrow \{1, 2, 3\}$

$A \longleftrightarrow 1$

$B \longleftrightarrow 2$

$C \longleftrightarrow 3$

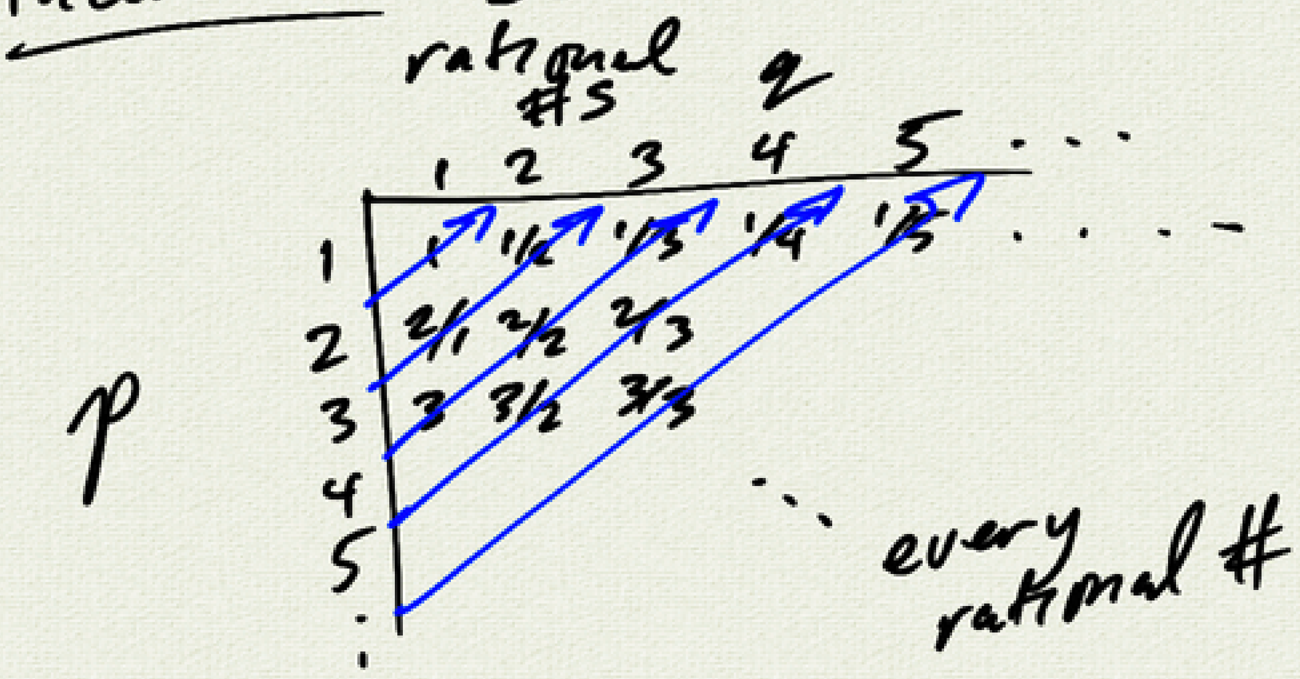
1-1 correspondence

a set X is *countable* if X is the size of $\mathbb{Z}$ integers (or counting #'s)

1	0
2	1
3	−1
4	2
5	−2
⋮	3
	−3
	⋮

all integers

Theorem: $\mathbb{Q}$ is countable



1
2/1
1/2
3
2/2
1/3
⋮

list of all rational #s

Theorem: $\mathbb{R}$ is not countable

$\mathbb{R}$ = real #s

Argue by contradiction:

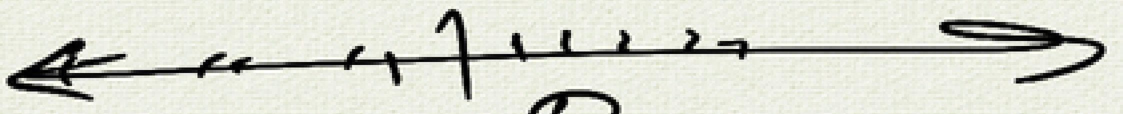
Assume $\mathbb{R}$ is countable

$\Rightarrow$ we can list all real #'s:

$.\boxed{1}2359$
$.9\boxed{1}234$
$.77\boxed{7}65$
$.561\boxed{2}3$
$.2341\boxed{2}$
⋮

$.22833,\ldots$ different from every # in the list

contradiction $\Rightarrow$ $\mathbb{R}$ is uncountable.



$\mathbb{Q}$
rational
countable
(finite expansion)

$\mathbb{R}$
real
uncountable
(irrational $\Rightarrow$ infinite expansion)

another countable set:
set of all finite binary sequences

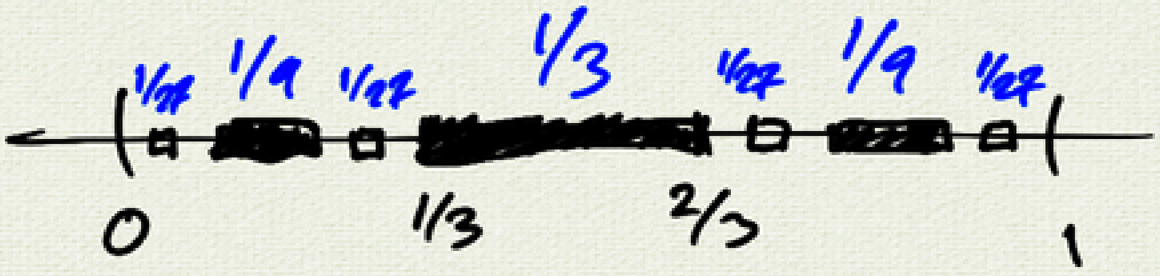
0
1
01
10
11
00
111
000
⋮

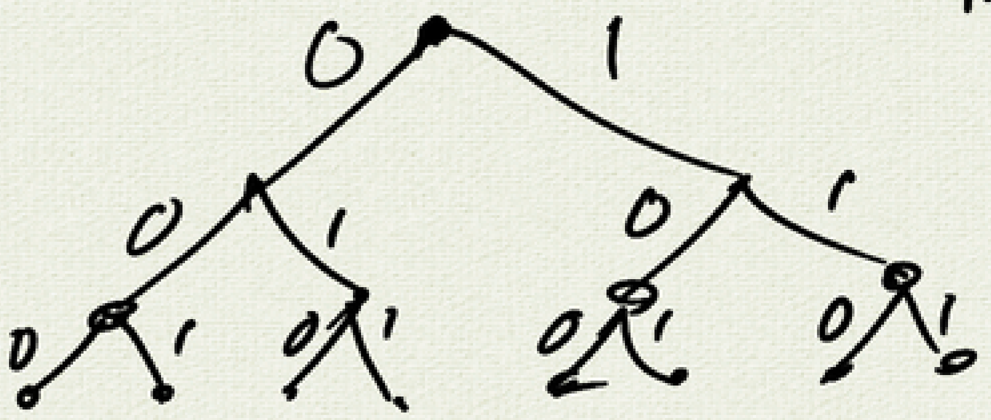
---

another uncountable set:
set of all infinite binary sequences

010101110...
10010...

Cantor Set





infinite binary sequence $\longleftrightarrow$ point in Cantor set

$\Longrightarrow$ Cantor Set is uncountable

what did I take out?

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots = \frac{a_1}{1-r} = \frac{1/3}{1-2/3} = 1$$

$a_1 = \frac{1}{3}$ $\quad r = \frac{2}{3}$

Cantor Set : uncountable set of measure zero

Q : countable, dense
