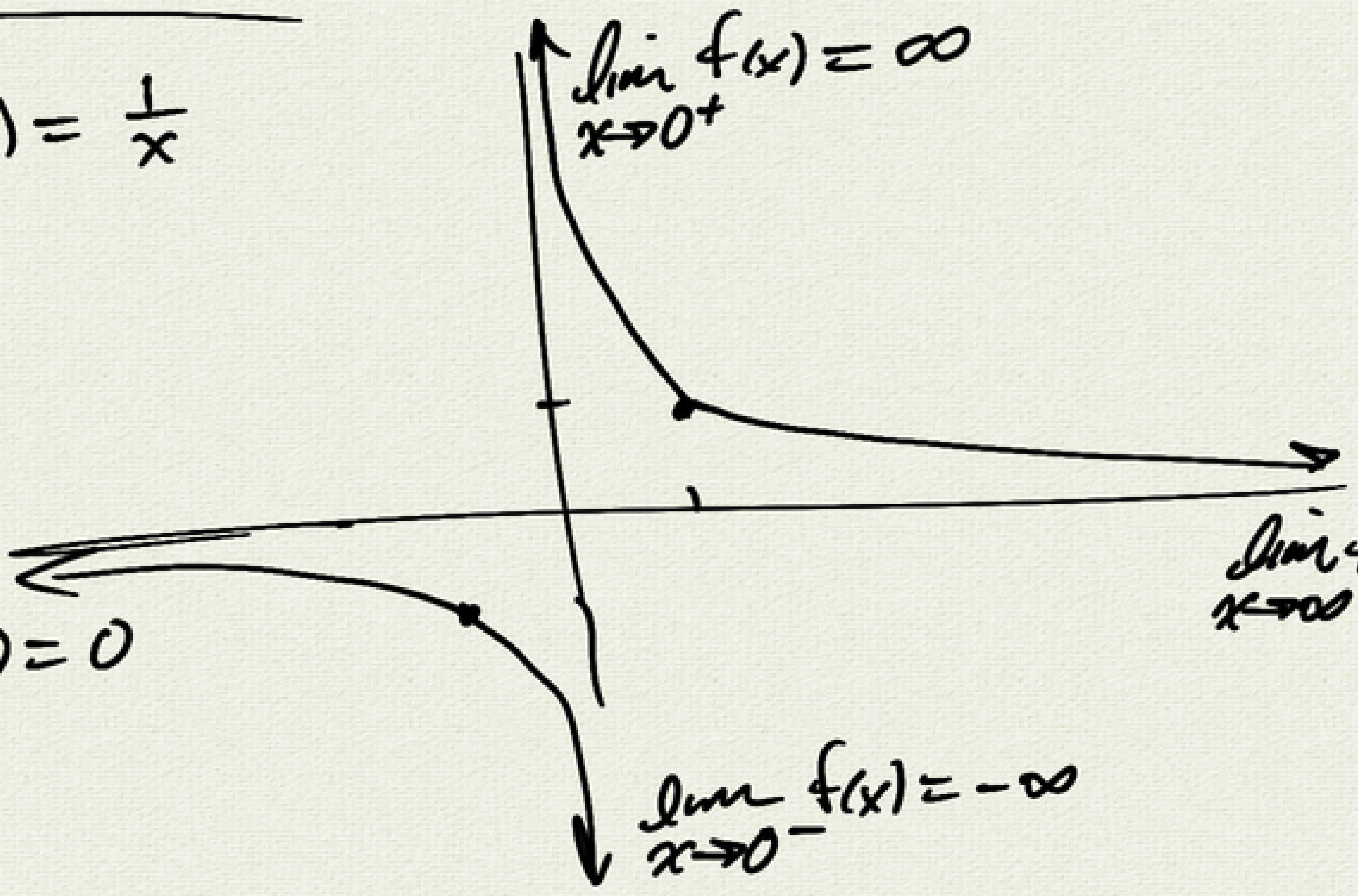
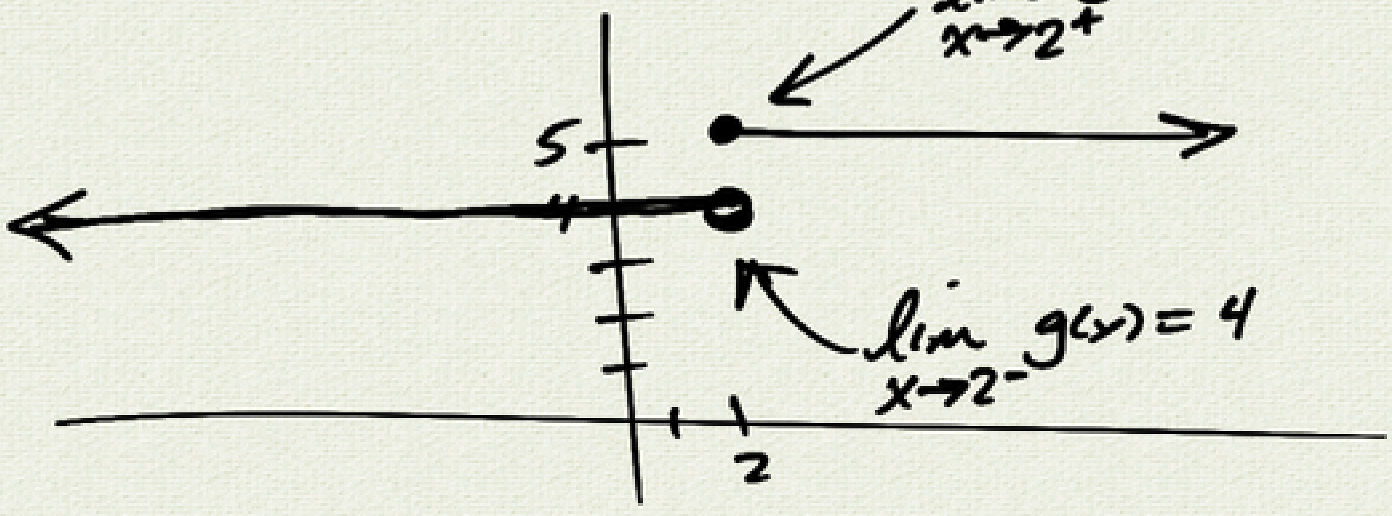
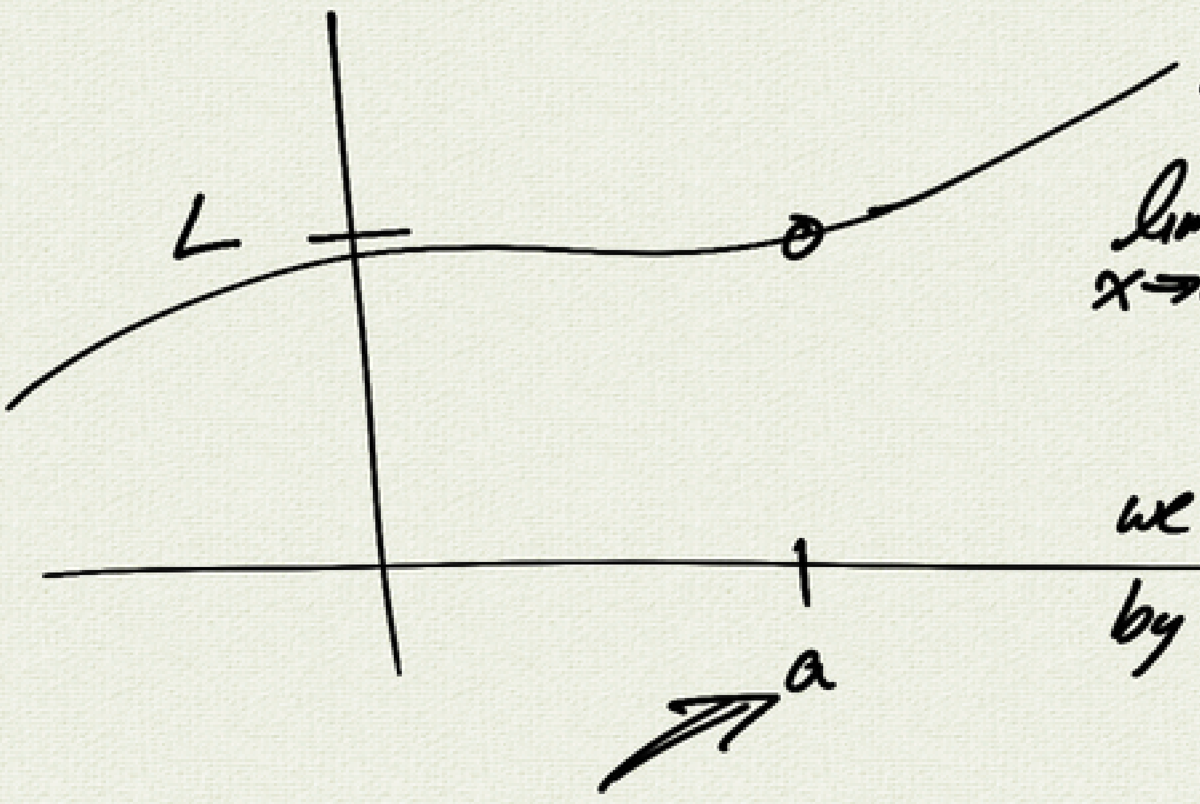
8.1 Limits

$f(x) = \frac{1}{x}$



$$g(x) = \begin{cases} 4 & \text{if } x < 2 \\ 5 & \text{if } x \geq 2 \end{cases}$$





define:

$\lim_{x \to a} f(x) = L$

means

we can make $f(x)$ close to L by taking x close to a

$\lim_{x \to a} f(x)$ exists $(=L)$ $\iff$ $\underbrace{\lim_{x \to a^-} f(x)}_{\text{left}} = \underbrace{\lim_{x \to a^+} f(x)}_{\text{right}} = L$

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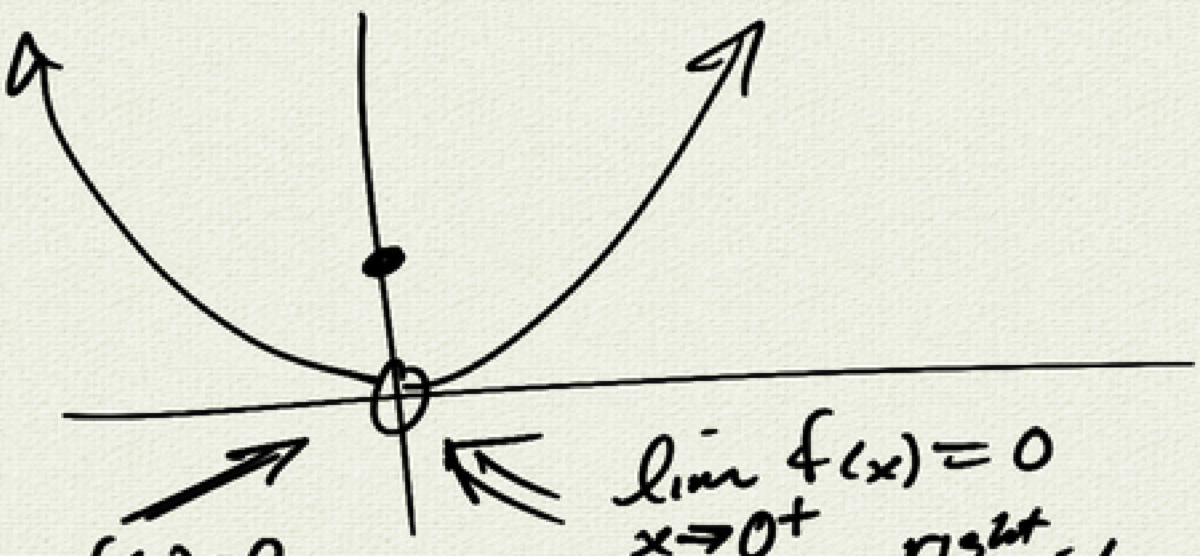
example:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

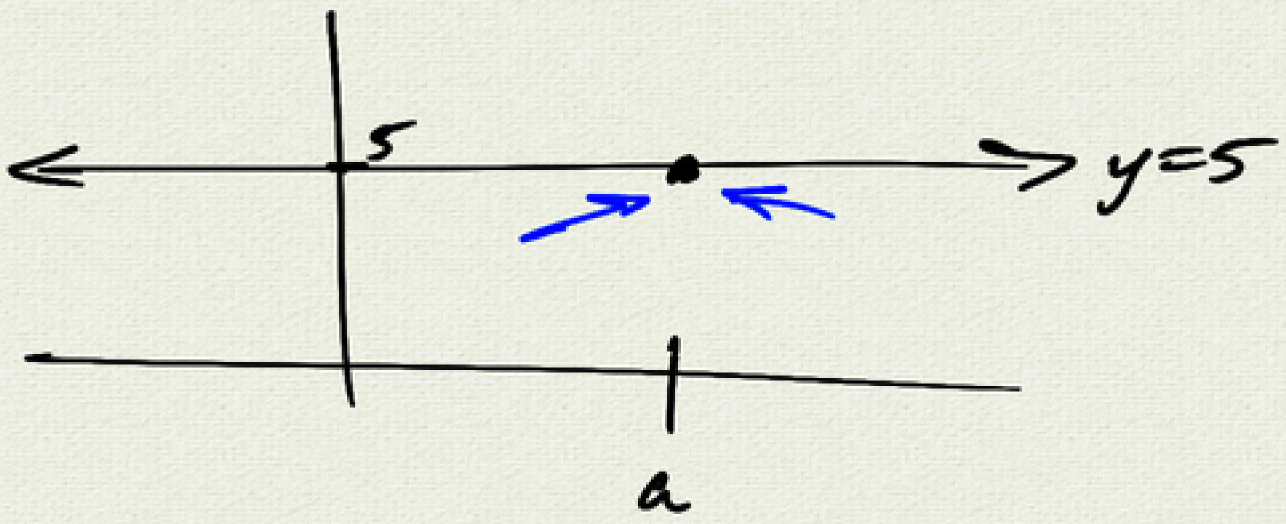


$\lim_{x \to 0^+} f(x) = 0$ right limit

$\lim_{x \to 0^-} f(x) = 0$ left limit

$\lim_{x \to 0} f(x) = 0$ exists (either side) observations

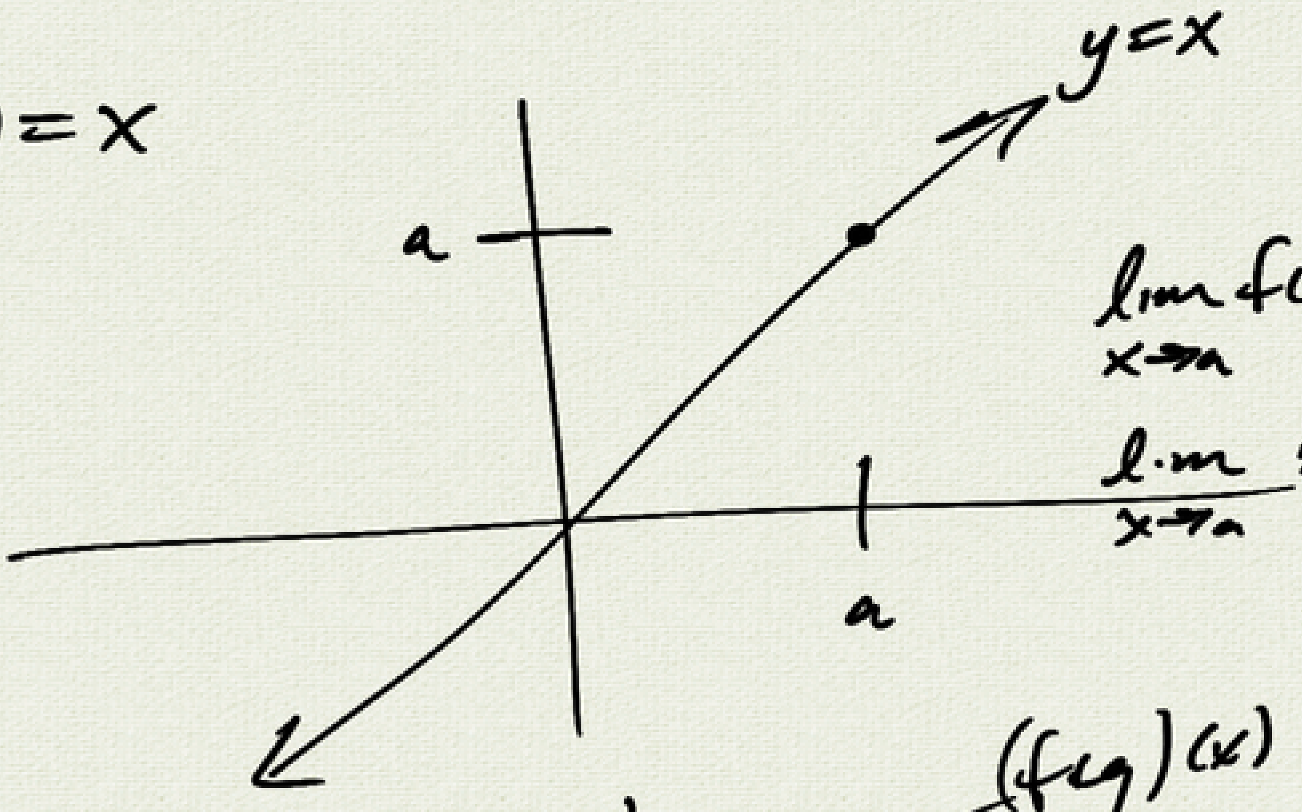
① $f(x) = 5$



$$\lim_{x \to a} f(x) = 5$$

$$\lim_{x \to a} (\text{const}) = \text{const.}$$

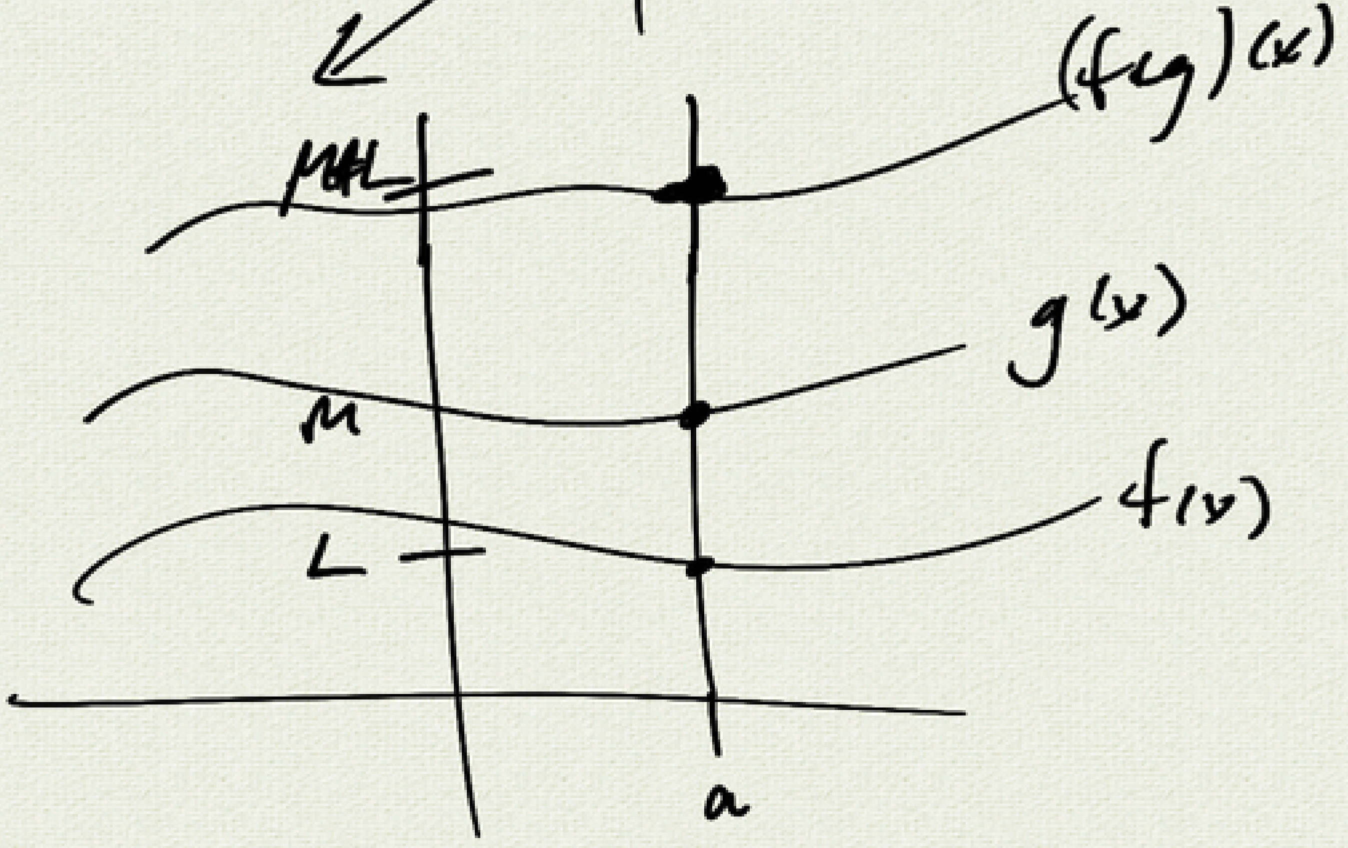
② $f(x) = x$



$$\lim_{x \to a} f(x) = a$$

$$\lim_{x \to a} x = a$$

③



suppose $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$ then $\lim_{x \to a} (f+g)(x) = L+M$

$$\lim_{x \to a} (fg)(x) = LM$$

and if $M \neq 0$

$$\lim_{x \to a} (f/g)(x) = L/M$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

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example:

$$g(x) = 2x^2 + 3x + 1$$

$$\lim_{x \to 2} g(x) = 2\left(\lim_{x \to 2} x\right)^2 + 3 \lim_{x \to 2} x + \lim_{x \to 2} 1$$

$$= \boxed{2(2)^2 + 3 \cdot 2 + 1}$$

$$= g(2)$$

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polynomials:

$$\lim_{x \to a} p(x) = p(a)$$

rational:

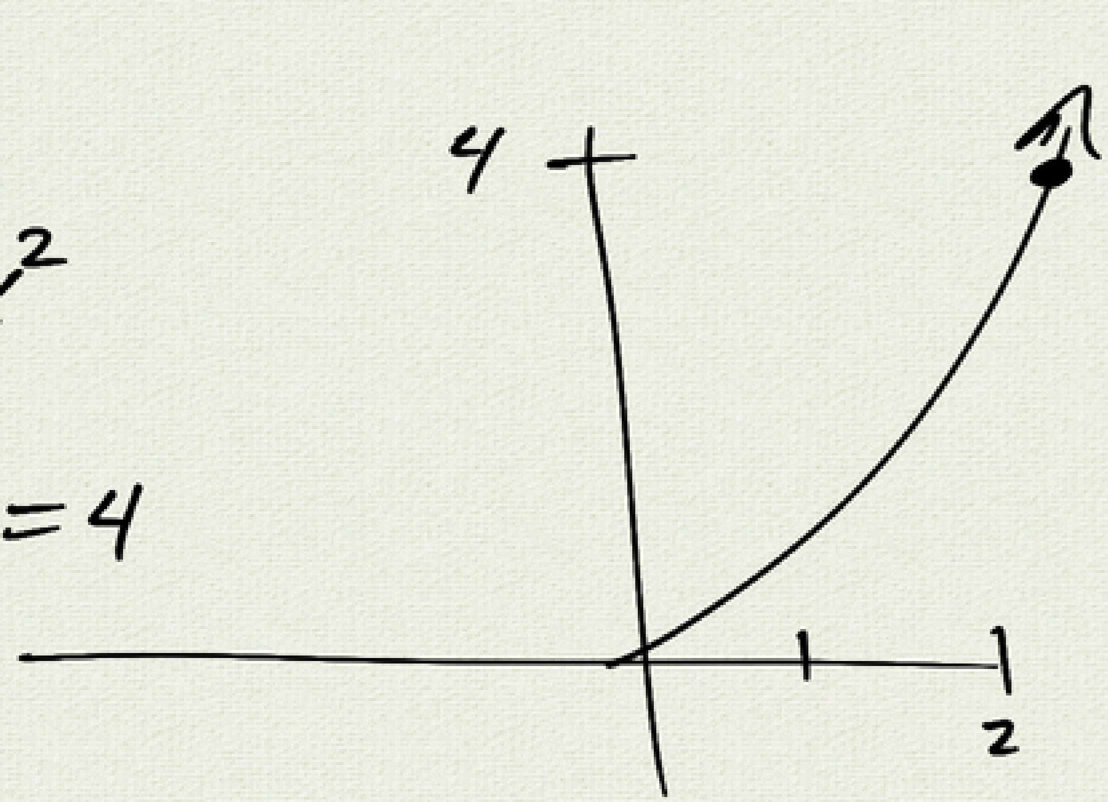
$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \leftarrow \text{as long as} \neq 0$$

true for:
trig functions
exp/log

example:

$$f(x) = x^2$$

$$\lim_{x \to 2} f(x) = f(2) = 4$$



example:

$$g(x) = \frac{x^2 - 4}{x + 2} = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}}$$

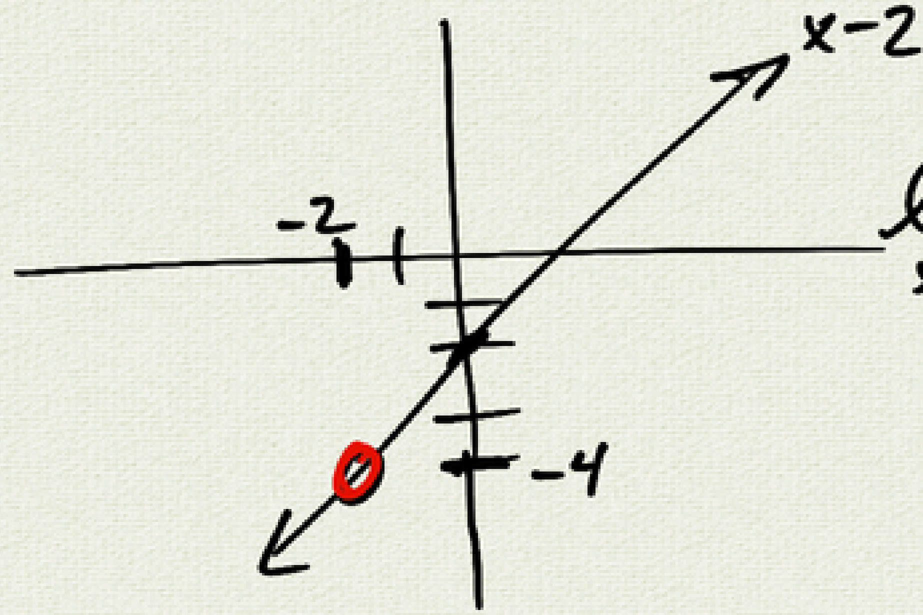
$$= \begin{cases} \boxed{x-2} & x \neq -2 \\ \text{undef} & x = -2 \end{cases} = \begin{cases} h(x) & x \neq -2 \\ \text{undef} & x = -2 \end{cases}$$

a=0

$$\lim_{x \to 0} g(x) = g(0)$$

$$= \frac{-4}{2}$$

$$= -2$$



$$\lim_{x \to -2} f(x) = h(-2)$$

a=-2

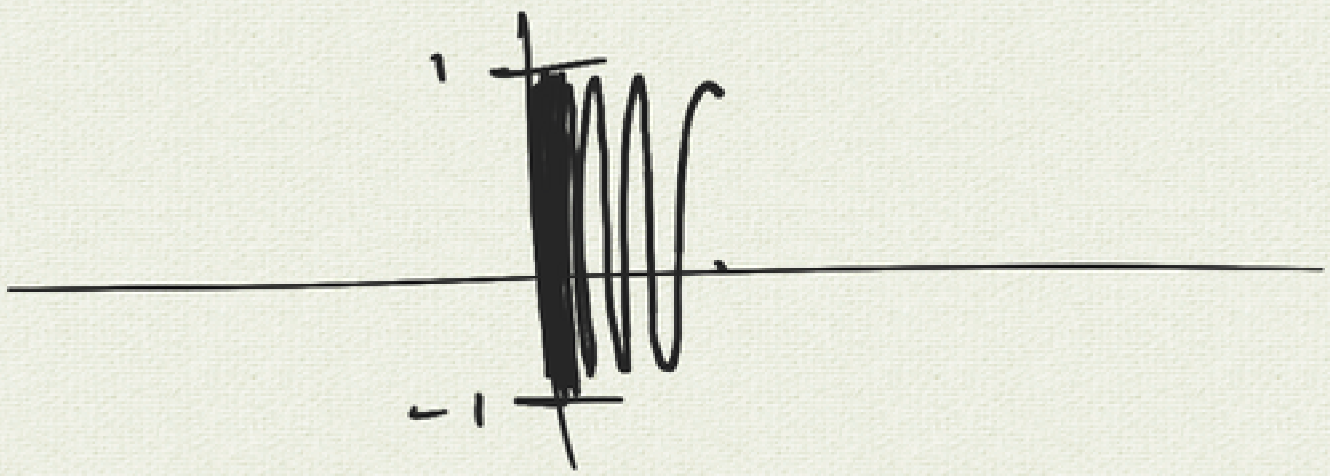
$$\lim_{x \to -2} g(x) = \boxed{\frac{0}{0}} \, ?$$

example:

$h(x) = \sin\boxed{\frac{1}{x}}$

$x \to 0$

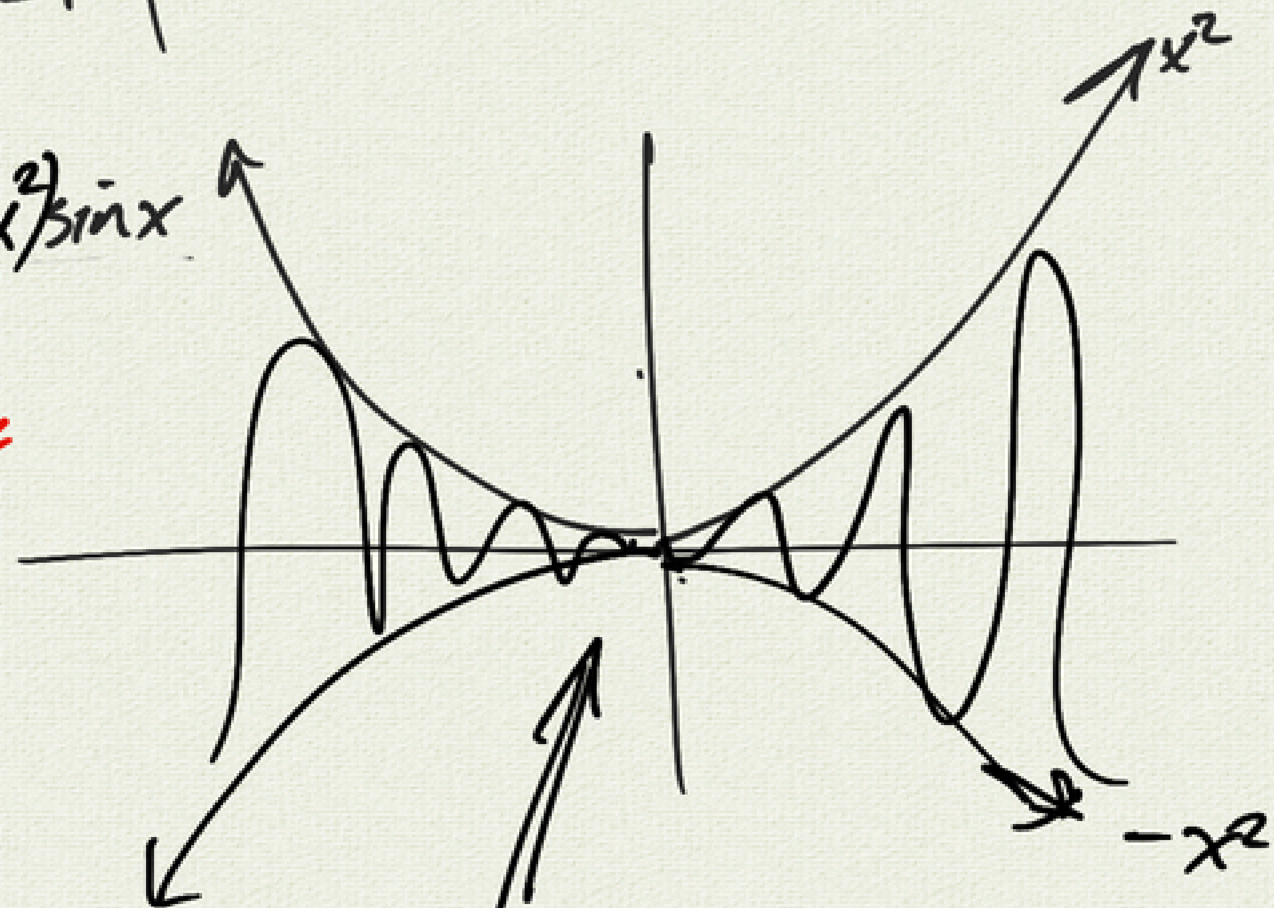
$\lim_{x\to 0^+} h(x)$ does not exist



example

$k(x) = (x^2)\sin x$

amplitude



$\lim_{x\to 0} k(x) = 0$

sandwich theorem
squeeze theorem

because:

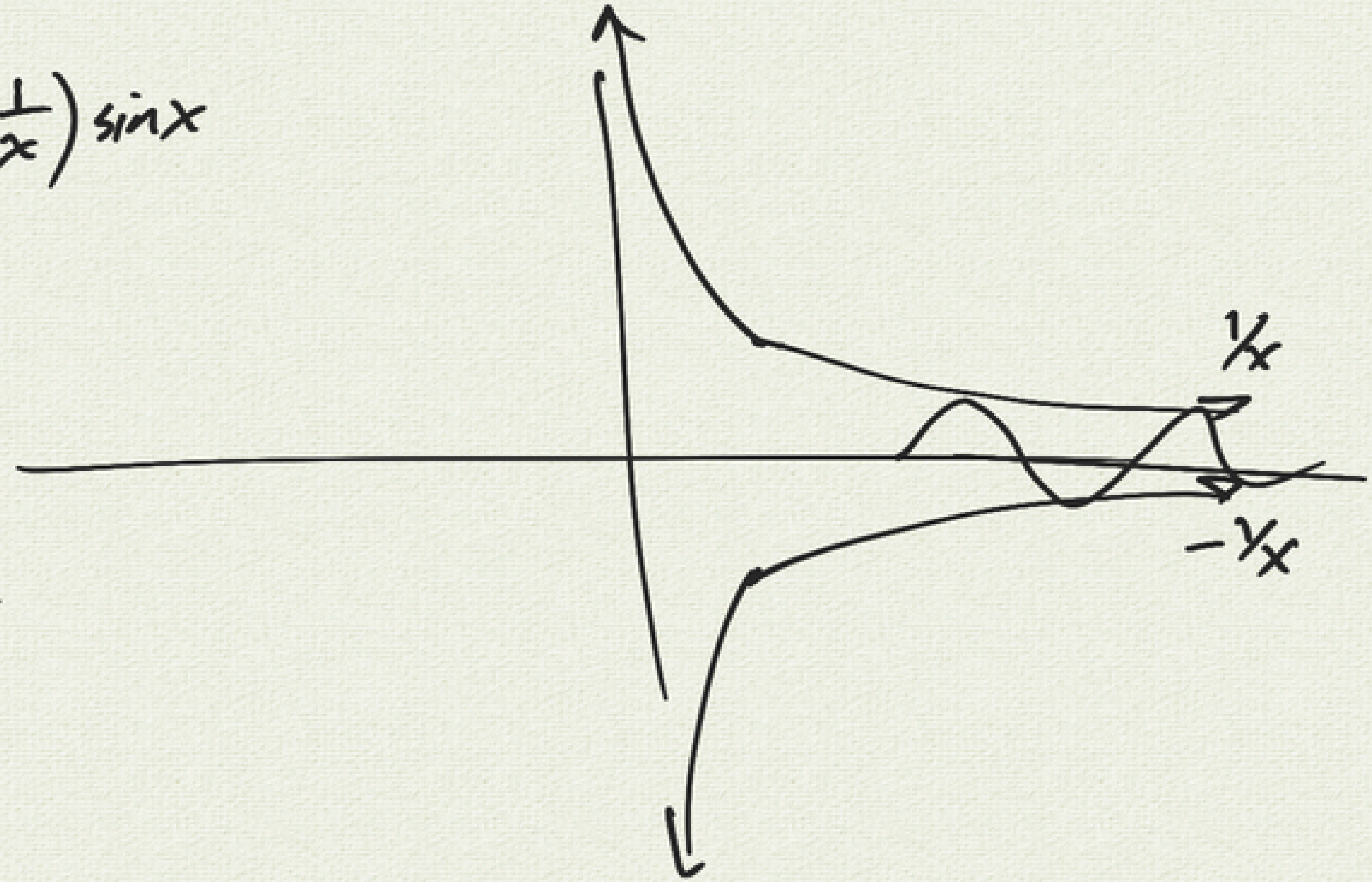
$-x^2 \leq k(x) \leq x^2$

and $\lim_{x\to 0} -x^2 = 0$

$\lim_{x\to 0} x^2 = 0$

$$f(x) = \frac{\sin x}{x} = \left(\frac{1}{x}\right)\sin x$$

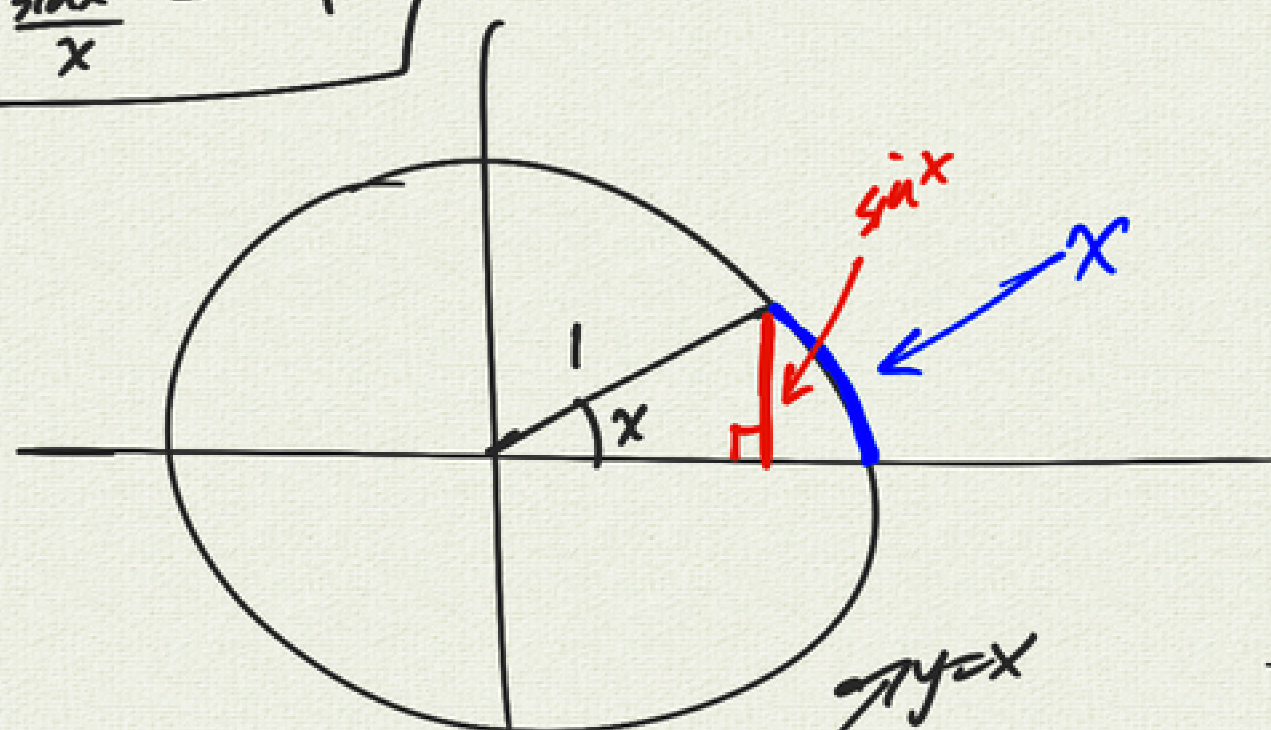
$x \to \infty$



$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x\to 0} \frac{\sin x}{x} = ?$$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$



as $x \to 0$:

$\sin x \approx x$

$\frac{\sin x}{x} \approx 1$

2 special limits:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}$$