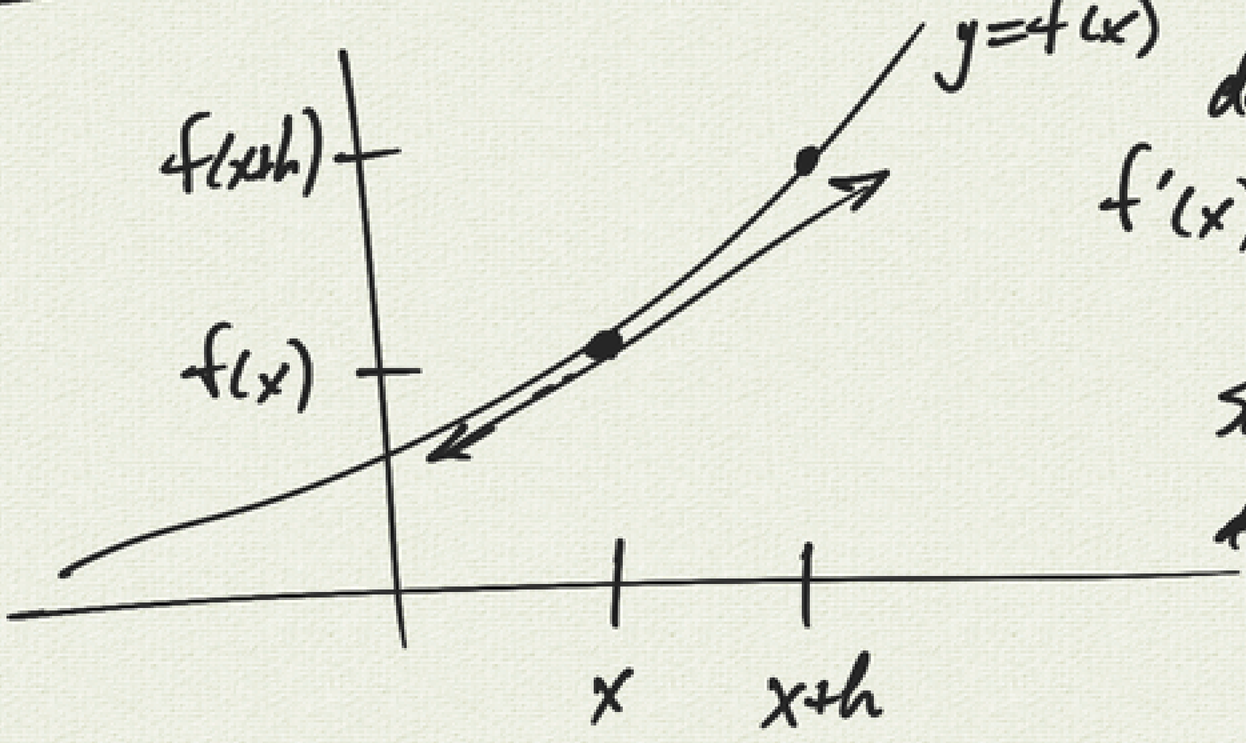
8.4 Derivative rules



define:

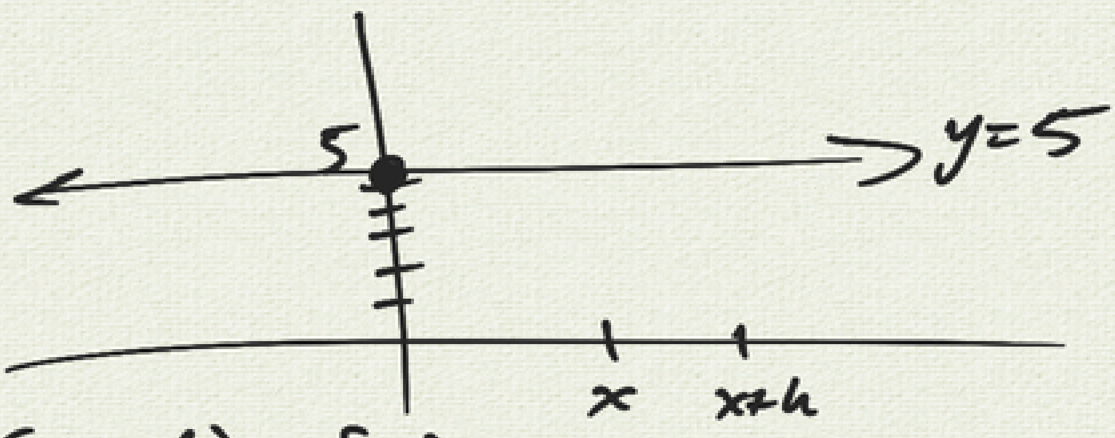
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

slope of tangent line

derivative of f at x

examples:

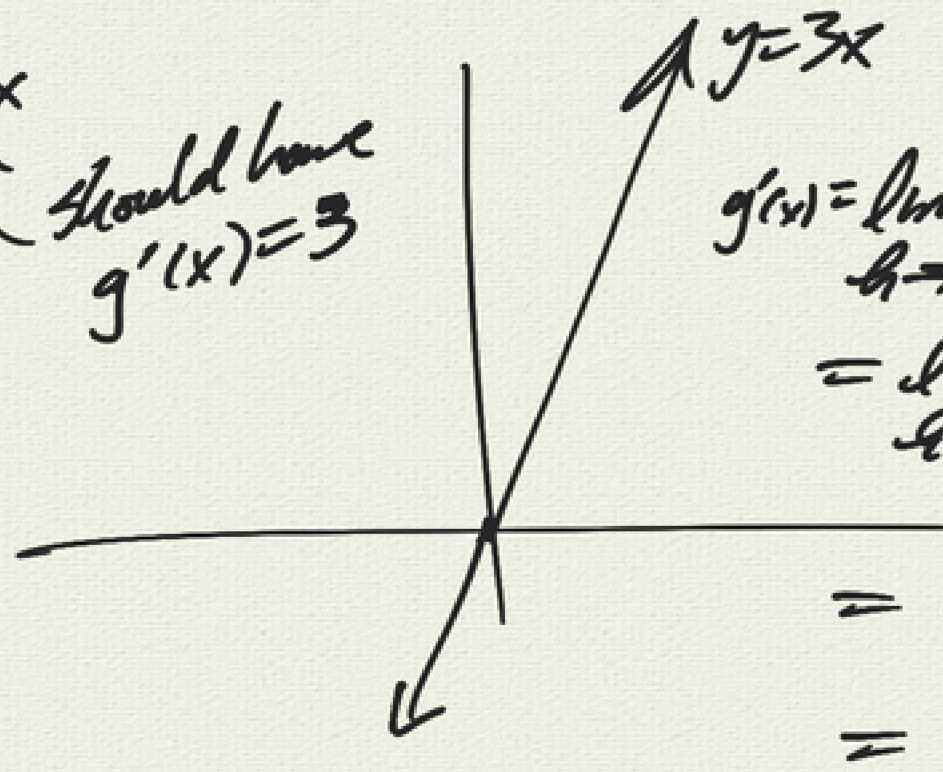
$f(x) = 5$



$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{5-5}{h}$$
$$= \lim_{h\to 0} 0$$
$$= 0 \Leftarrow \text{makes sense}$$

$g(x) = 3x$

should have $g'(x) = 3$



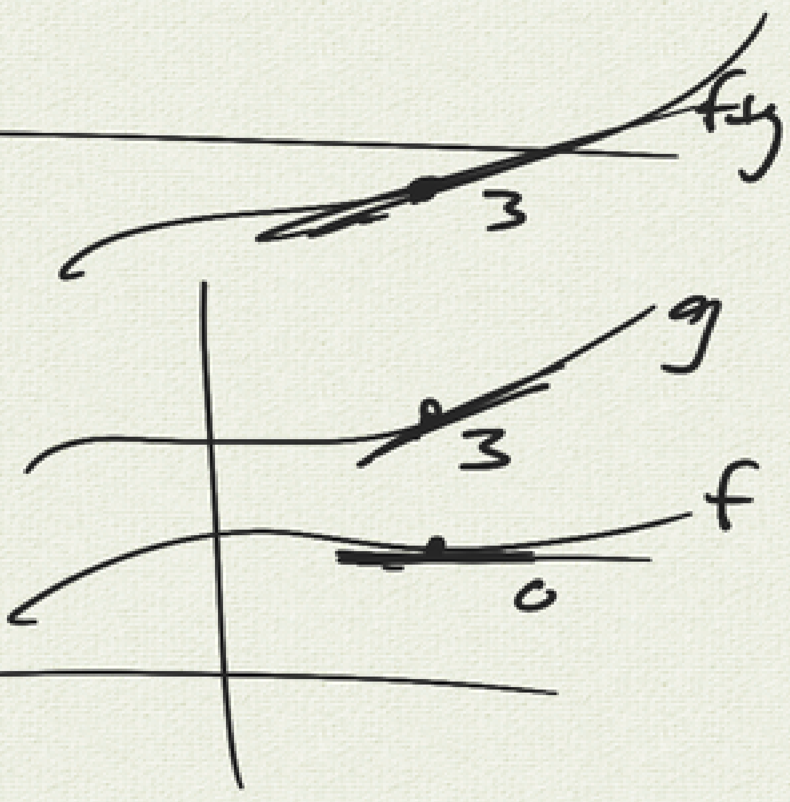
$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
$$= \lim_{h\to 0} \frac{3(x+h) - 3x}{h}$$
$$= \lim_{h\to 0} \frac{3h}{h}$$
$$= 3$$

what if $g(x) = 3x+5 \Rightarrow g'(x) = 3$

$$\boxed{(f+g)'(x) = f'(x) + g'(x)}$$

sum rule

subtraction too



$(f+g)'(x)$
$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

other notation: $f'(x) = \frac{df}{dx} = \boxed{\frac{dy}{dx}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

sum formula: $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

think: $dx \approx$ really small Δx

infinitessimal

$f(x) = x^n$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \quad \rightarrow \frac{0}{0}\ ?$$

$$= \lim_{h \to 0} \frac{\left(x^n + \overset{\binom{n}{1}}{n} x^{n-1} h^1 + \binom{n}{2} x^{n-2} h^2 + \ldots + h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} \frac{n x^{n-1} h}{h} + \underbrace{\frac{\binom{n}{2} x^{n-2} h^2}{h}}_{\rightarrow 0} + \underbrace{\frac{\binom{n}{3} x^{n-3} h^3}{h}}_{\rightarrow 0} + \ldots$$

$$\boxed{\frac{d}{dx}\left(x^n\right) = n x^{n-1}}$$

power rule

<u>example</u>: $f(x) = x^2 \Rightarrow f'(x) = 2x$

$g(x) = x^3 \Rightarrow g'(x) = 3x^2$

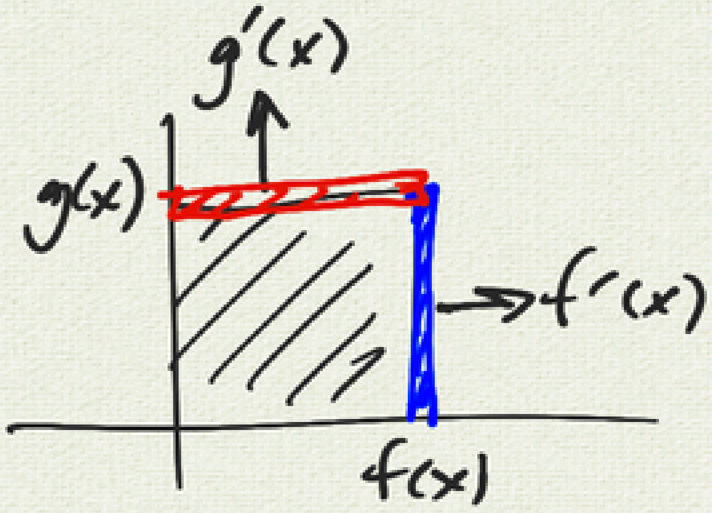
$f(x) = 5x^2 \Rightarrow f'(x) = 10x$

$$\frac{d}{dx}\left(k f(x)\right) = k\left(\frac{df}{dx}\right)$$

$p(x) = x^5 + 4x^3 + 3x^2 + x + 1$

$$\Rightarrow \boxed{p'(x) = 5x^4 + 12x^2 + 6x + 1}$$

product: $(fg)'(x) = ?$



$$A(x) = f(x)g(x)$$
$$A'(x) = \underline{f'(x)g(x)} + \underline{f(x)g'(x)}$$

$$\boxed{\text{product rule: } (fg)'(x) = f'(x)g(x) + f(x)g'(x)}$$

$$\frac{d}{dx}(fg) = \left(\frac{df}{dx}\right)g + f\left(\frac{dg}{dx}\right)$$

slope = rate of change

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example:

$$f(x) = (2x+1)(3x^2+5)$$

2 ways:

① multiply first:

$$f(x) = (2x+1)(3x^2+5)$$
$$= 6x^3 + 3x^2 + 10x + 5$$
$$\Longrightarrow f'(x) = 18x^2 + 6x + 10$$

$\frac{d}{dx}(2x+1) = 2$ $\quad$ $\frac{d}{dx}(3x^2+5) = 6x$

② product rule:

$$f'(x) = 2(3x^2+5) + (2x+1)(6x)$$
$$= 6x^2 + 10 + 12x^2 + 6x$$
$$= 18x^2 + 6x + 10 \quad \checkmark$$

question: $f(x) = \frac{1}{x} = x^{-1}$

power rule:
$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) \stackrel{?}{=} -1\,(x)^{-2}$$
$$= -\frac{1}{x^2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x - (x+h)}{x(x+h)}\right]$$
$$= \lim_{h \to 0} -\frac{h}{h}\,\frac{1}{x(x+h)}$$

$$\boxed{f'(x) = -\frac{1}{x^2}}$$ $\Leftarrow$ power rule works for negative integers too

$$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0$$ $\Leftarrow$ and 0

differentiation $\Longleftrightarrow$ finding the derivative

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quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'g - fg'}{(g(x))^2}$

Summary :

$\frac{d}{dx}(\text{const}) = 0$ | $f(x)=\text{const} \rightarrow f'(x)=0$

$\frac{d}{dx}(mx) = m$ | $f(x)=mx \rightarrow f'(x)=m$

$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$ sum | $(f+g)'(x) = f'(x)+g'(x)$

$\frac{d}{dx}(kf) = k\frac{df}{dx}$ scalar multiple | $(kf)'(x) = kf'(x)$

$\frac{d}{dx}(x^n) = nx^{n-1}$ power rule | $f(x)=x^n \rightarrow f'(x)=nx^{n-1}$

$\frac{d}{dx}(fg) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$ product rule | $(fg)'(x) = f'g + fg'$

$= f'(x)g(x) + f(x)g'(x)$